GEOMETRICAL
DRAWING

PART. II.

WINTER

ELEMENTARY

GEOMETRICAL DRAWING.

PART II.

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ELEMENTARY

GEOMETRICAL DRAWING.

PART II.

THE PRACTICAL GEOMETRY OF PLANES AND SOLIDS:

COMPRISING THE

ELEMENTS OF DESCRIPTIVE GEOMETRY, WITH
ITS APPLICATION TO HORIZONTAL AND ISOMETRIC PROJECTION,
AND THE PROJECTION OF SOLIDS AND SHADOWS.

CHIEFLY DESIGNED FOR
THE USE OF STUDENTS PREPARING FOR MILITARY EXAMINATIONS.

BY

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1861.

PREFACE.

THE object of the present volume, and of Part I., is to furnish students with an easy introduction to Geometrical Drawing, from which they may acquire an amount of knowledge sufficient to enable them to solve such questions as are usually set, in this branch, at Military Examinations; and, at the same time, to qualify them to enter upon the study of the more difficult portions of the subject.

The writer has been compelled by constant occupation to allow a long interval to elapse between the publication of the First and Second Parts; the success of the former, in the meantime, and the numerous inquiries addressed to him respecting the probable appearance of the latter, have confirmed him in the opinion that such a work was needed.

The principal authors consulted in the preparation of the work were: MONGE; LEFÉBURE DE FOURCY; LEROY; LE BLANC; REYNAUD; LAMBERT ET PICQUÉ; and from them great assistance was derived.

The Exercises, of which a numerous collection is appended to each Chapter, have been entirely selected from the Woolwich Papers.

WOODFORD, London, N.E.
July, 1861.

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PRACTICAL

GEOMETRY OF PLANES AND SOLIDS.

Errata.

In the references, *for* (33) *read* (34), and *for* (35) *read* (36).
Page 89, line 5, *for* above *read* about.
" 91, " 9, " sides " ends.

2. *Def.*—The given planes are called the PLANES OF PROJECTION.

3. *Def.*—The PROJECTION OF A POINT is the foot of the perpendicular, drawn from the point to the plane of projection.

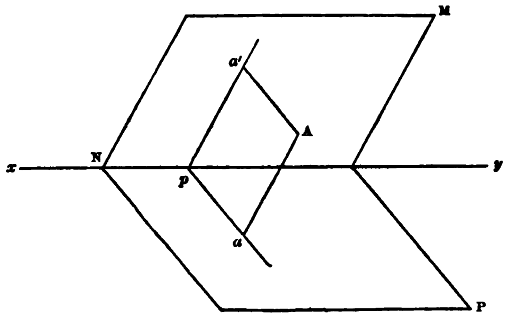
4. *Def.*—This perpendicular is called the PROJECTOR of the point.

If A (Fig. 1) be a point in space, M N and N P the planes of projection, cutting each other in xy, and A a be drawn perpendicular to the plane N P, a will be the projection of A on N P; A a, its projector. Similarly A a' and a' are its projector and projection, with reference to the plane M N.

5. *Def.*—When the projectors are perpendicular to the planes of projection the projection is styled ORTHOGRAPHIC.

6. *Def.*—The PROJECTION OF A LINE is the line which passes through the projections of all points in that line.

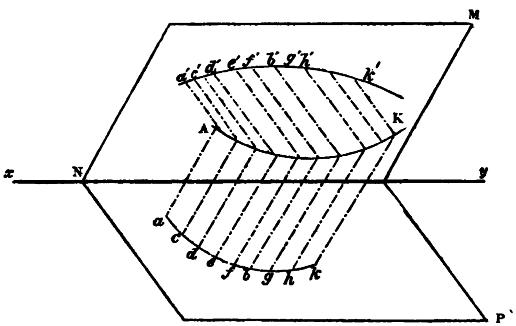
Fig. 1.



7. *Def.*—The PROJECTING SURFACE OF A LINE is the surface, plane or otherwise, which contains the projectors of all points in that line.

If A B (Fig. 2) be any line in space *a*, *c*, *d*, *e*, *f*, *b*, &c., the projections of its several points on the plane N P, the line

Fig. 2.



a c d e f b g h k, which passes through all these points, will be the projection of A B on the plane N P, and the surface A *a k* K will be its projecting surface.

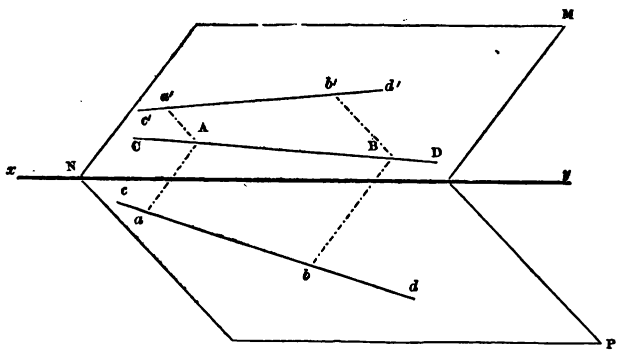
8. *Theorem.*—The projection of a straight line will be a straight line.

For the projectors of all its points being drawn from points in a straight line, and being also perpendicular to the plane of projection, will lie in one plane. (*Euc.* XI. 6, and I. *def.* 35.) The intersection of this plane with the plane of projection is a line which passes through the projections of all points in the given line, it is therefore the projection of that line (6), and it is a straight line (*Euc.* XI. 3).

Cor. It is evident that the projecting surface of a straight line is a PLANE. Also that the projection of a straight line is determined by the projections of any two points in that line, since only one straight line can pass through the same two points.

In Fig. 3, let a and b be the projections of A and B, any two points in the indefinite straight line C D in space; the straight

Fig. 3.



line $c\,d$, passing through a and b, will be the projection of C D, on the plane N P.

9. Since A a and B b are perpendicular to N P, the plane A a b B is also perpendicular to N P (*Euc.* XI. 18). Consequently the projecting plane of a straight line is the plane containing the line, and perpendicular to the plane of projection.

10. If a straight line be perpendicular to the plane of projec-

tion, its projection will be a point. This is evident, because the projectors of all points in the line will coincide with the line itself.

11. If two straight lines be parallel their projections upon the same plane will be parallel (*Euc.* XI. 16). In order, therefore, to construct the projections of two parallel lines, it will be sufficient to know the projections of two points in one of them, and of one point in the other.

12. When the line projected is not a straight line, its projecting surface will generally be what is termed a cylindrical one, and its projection a curve. If, however, the curve is a plane curve, and its plane is perpendicular to the plane of projection, its projection will manifestly be a straight line.

The projection of a line will always contain the projections of all points and lines on its projecting surface.

13. If a line be situated in a plane parallel to the plane of projection, its projection will be equal to the line itself. In the case of a straight line this is evident from *Euc.* I. 33. In the case of a curve, the line and its projection may be considered as the intersections of a cylinder by two parallel planes.

14. *Theorem.*—If two planes cut each other, and a point be projected on both of them, the perpendiculars drawn from the projections to the intersection of the planes will meet that intersection in the same point.

Let the planes M N and N P (Fig. 1) cut each other in $x\,y$.

Let a, a' be the projections of a point, A, in space upon N P and M N respectively.

Then the plane a' A a will be perpendicular to M N and N P (*Euc.* XI. 18), and will cut $x\,y$ in some point p; consequently (*Euc.* XI. 19), $x\,y$ is perpendicular to the plane a A a'; the angles $y\,p\,a$ and $y\,p\,a'$ are therefore right angles (*Euc.* XI. def. 3). Now the straight lines $p\,a$ and $p\,a'$ pass through a and a' the projections of A; therefore the perpendiculars drawn from a and a' must meet $x\,y$ in the same point.

Conversely. When two points, one in each plane of projection,

are so situated that the perpendiculars drawn from them to the intersection of those planes meet the intersection in the same point, these two points may be considered as the projections of a single point in space.

15. A straight line is generally determined by its projections on two given planes, since the line itself is the intersection of its projecting planes. When however the projections are perpendicular to the intersection of the planes, a third plane of projection will be necessary to determine the line, since the projections may be those of any line in a plane perpendicular to both planes of projection.

16. Two straight lines assumed arbitrarily, one in each plane of projection, can only be considered as the projections of the same straight line in space, when the planes containing those lines, and perpendicular respectively to the planes of projection, are not parallel. Similarly two curves, one in each plane of projection, can only be considered as the projections of the same curve in space, when the cylindrical surfaees, passing through these curves, and perpendicular respectively to the planes of projection, cut each other. The curve itself will be the intersection of these surfaces.

17. *Def.*—THE TRACE OF A LINE is the point in which it meets the plane of projection.

18. *Def.*—THE TRACE OF A PLANE is the line in which it cuts the plane of projection.

19. A plane is determined by its traces, since only one plane can pass through the same two straight lines.

20. *Def.*—The intersection of the planes of projection is called the GROUND LINE or AXIS.

21. If a plane be not parallel to the ground line it will meet it in a point common to both of its traces.

22. If a plane be parallel to the ground line its traces will also be parallel to the ground line.

This is evident, for since the axis is parallel to the plane it cannot meet it, and consequently cannot meet the traces which are lines in the plane; but each trace and the axis are in one plane, therefore they are parallel (*Euc.* I. Def. 35).

If a plane be perpendicular to the axis, its traces will be perpendicular to the axis (*Euc.* XI. 19, and Def. 3).

If a plane be parallel to one plane of projection its trace upon the other will be parallel to the ground line (*Euc.* XI. 16).

If a plane contain the axis, a third plane of projection will be necessary to determine it.

23. No reference has yet been made to the magnitude of the angle contained by the planes of projection. In what follows this angle will be assumed to be a right angle, since that supposition tends to simplify the constructions employed in the solution of problems.

For the sake of distinction, one plane of projection will be called the VERTICAL PLANE, the other the HORIZONTAL.

24. *Def.*—Traces and projections are styled VERTICAL or HORIZONTAL accordingly as they are situated in the vertical or in the horizontal plane. A vertical projection is sometimes called an ELEVATION, and a horizontal projection is called a PLAN.

25. *Def.*—A HORIZONTAL LINE is a line parallel to the horizontal plane.

26. *Def.*—A VERTICAL LINE is a line perpendicular to the horizontal plane.

27. The following consequences result from assuming the planes of projection at right angles to each other.

I. The projections of all points in the planes of projection are in the ground line.

II. The projections of all lines situated in a plane parallel to one of the planes of projection are parallel to the ground line.

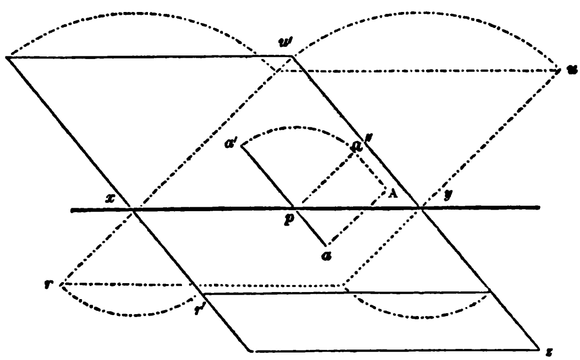
III. If a plane be perpendicular to one plane of projection, and not parallel to the other, its trace upon that other is perpendicular to the ground line. (*Euc.* XI. 19, and Def. 3.)

IV. The distance of the elevation of a point from the ground line shows the distance of the point from the horizontal plane: the distance of the plan of a point from the ground line shows the distance of the point from the vertical plane.

28. From the foregoing explanations it might be inferred that two planes were necessary for the purpose of representing plans and elevations in their real magnitude. Such, however, is not the case. For all constructions required may be united in one drawing and on a single plane, by supposing one of the planes of projection to revolve about the ground line until it coincides with the other plane of projection. In practice it is usual to turn the vertical plane back about the ground line, until it coincides with the horizontal plane.

Thus, in Fig. 4, the vertical plane, $x\,y\,u$, revolves about $x\,y$ until it takes the position $x\,y\,u'$, when $x\,y\,z$, the horizontal plane,

Fig. 4.



and $x\,y\,u'$ form one plane. All constructions will now be made in the horizontal plane, and in order to arrive at a correct idea of the relative positions of points, given by their projections only, the vertical plane must be conceived to take its original position. For example, if a, a' be the projections of a point A in space, Fig. 4, and the plane $x\,y\,u'$ be supposed to make one-fourth of a revolution about $x\,y$, so as to take the position $x\,y\,u$;

and from a and a'' two straight lines be drawn perpendicular to $x\,y\,z$ and $x\,y\,u$ respectively, these perpendiculars will intersect in a point which will be A.

29. After the vertical plane has been turned down to coincide with the horizontal plane, that portion of it which was below the ground line, as $x\,y\,r$, will take the position $x\,y\,r'$. Any elevations on it will therefore be in front of the ground line.

It is evident that points in space will be represented by their projections only.

30. *Theorem.*—The plan and the elevation of a point are both situated in a straight line perpendicular to the axis.

This follows at once from (14); for during the revolution of $x\,y\,u$ about $x\,y$ (Fig. 4), $p\,a''$ remains perpendicular to $x\,y$; therefore (*Euc.* I. 14), $a\,a'$ is a straight line; and it is perpendicular to $x\,y$.

31. *Theorem.*—If a straight line be perpendicular to a plane its projections will be respectively perpendicular to the traces of that plane.

For, the trace of the given plane is the line in which it cuts the plane of projection (18);

The projecting plane of the line is perpendicular to the plane of projection (9), and also to the given plane (*Euc.* XI. 18).

The trace is therefore perpendicular to the projecting plane (*Euc.* XI. 19); and

Consequently, to the projection of the line (*Euc.* XI. Def. 3.).

32. *Def.*—A DIHEDRAL ANGLE is the angle contained by two intersecting planes.

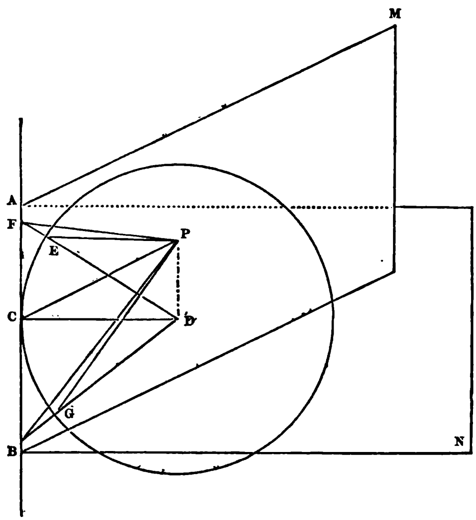
33. *Def.*—THE PROFILE ANGLE of two planes is the angle contained by the two straight lines in which these planes are cut by a third plane, at right angles to both of them. This third plane is called a PROFILE PLANE. Since these lines are perpendicular to the intersection of the two given planes (*Euc.* XI. 19, and Def. 3),

the profile angle will be the measure of the dihedral angle (*Euc.* xi. Def. 6).

34. *Theorem.*—The profile angle of two planes is the greatest acute angle contained by any two straight lines, one in each plane, which meet each other.

Let A B (Fig. 5) be the intersection of two planes, M A B, N B A, P C D their profile angle (33). In A B take any point

Fig. 5.



F, draw P D perpendicular to C D, it will be perpendicular to the plane N B A (*Euc.* xi. Def. 4), join P F and D F, make D E equal to D C, and join P E. Then (*Euc.* i. 4) the angle P C D is equal to the angle P E D; but the angle P E D is greater than the angle P F D (*Euc.* i. 16), therefore the angle P C D is greater than the angle P F D, which is the inclination of a line P F in the plane M A B to a line D F in the plane N B A; and the same may be proved of any other such line.

Cor. 1. Since P F D is the inclination of the line P F to the plane N B A (*Euc.* xi. Def. 5), it is evident that no line in

the plane M A B can make with the plane N B A an angle greater than the angle between the planes.

Cor. 2. Let E C G be a circle, whose centre is D and radius D C, then, since the angle P C D is equal to the angle P E D equal to the angle P G D, it is evident that all straight lines passing through a given point, and making a given angle with a given plane, meet that plane in the circumference of a circle whose centre is the foot of the perpendicular drawn from the point to the plane.

Cor. 3. The projection of the line P C on the plane N B A is the line C D, which, by plane trigonometry, is equal to the line P C multiplied by the cosine of the angle P C D. If, therefore, L be the length of a straight line, θ the angle at which it is inclined to the plane of projection, P the projection,

$$P = L \times \cos \theta.$$

35. The *Notation* employed in Descriptive Geometry should be simple and uniform. In the following problems points in space will be denoted by italic capitals, as A, B, C; the plans of these points by the corresponding small letters, as a, b, c; their elevations by the same small letters distinguished by an accent, thus, a', b', c'.

The point (a, a') will therefore denote the point A, in space, whose plan is a, and elevation a'.

The line $(a\,b,\ a'\,b')$ will denote the line $A\,B$, in space, whose plan and elevation are $a\,b$ and $a'\,b'$ respectively.

36. It has been shown (28) that the constructions in Descriptive Geometry are all brought into one plane by supposing the vertical plane of projection to revolve about the ground line until it coincides with the horizontal plane. A similar process, viz. turning any plane about one of its traces until it coincides with the plane containing that trace, often facilitates the solution of Problems.

This method of solving problems (called by French writers *La méthode des Rabattements*) depends upon the following principle.

After a plane has been turned about one of its traces until it

coincides with the plane of projection containing that trace, any point in the plane will be situated in a straight line drawn through the projection of the point, perpendicular to the trace. The distance of such point from the trace will be equal to the hypothenuse of a right angled triangle, whose base is the distance of the projection from the trace, and whose perpendicular is the distance of the point from the plane of projection containing the trace.

This will appear evident if a profile plane be drawn through the point, for then the projector of the point will be the perpendicular of the triangle, the intersection of the profile plane with the plane of projection will be its base, and the intersection of the profile plane with the given plane will be its hypothenuse. After the revolution of the plane this hypothenuse will either coincide with the base, or be in the same straight line with it, and the point in question will be at the extremity of the hypothenuse in its new position.

The application of this method to a given plane will now be explained.

I. When the given plane is perpendicular to both of the planes of projection.

II. When it is perpendicular to one of them only.

III. When it is perpendicular to neither of them.

I. Let (a, a') (Fig. 6) be the point situated in the plane M N P, which is perpendicular to both of the planes of projection. The traces M N and N P will form a straight line perpendicular to $x y$.

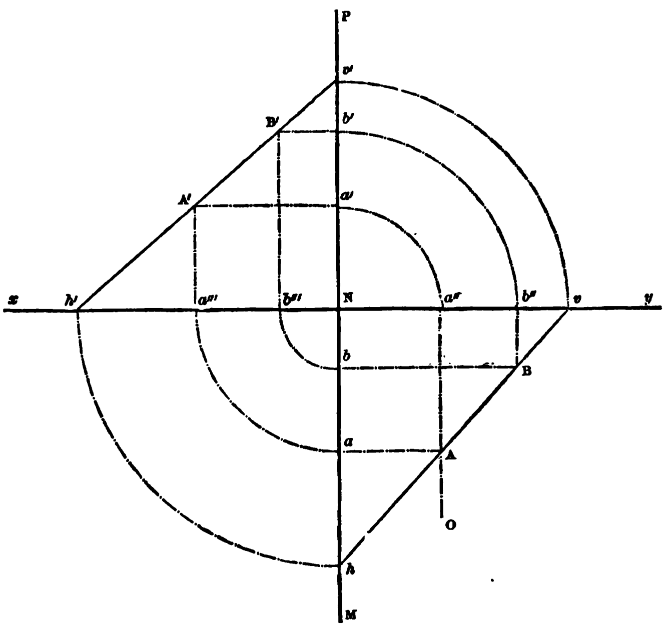
In this case, when the plane has been turned about its horizontal trace M N, the distance of the point A from that trace will evidently be equal to N a'. If, therefore, with N as a centre, and a radius N a', a circle $a' a''$ be described, cutting $x y$ in a''; a'' O be drawn perpendicular to $x y$, and a A perpendicular to M N, A will be the point required. In the same manner a second point B (b, b') may be found, and thus the straight line A B containing these points is determined.

By a similar construction A$'$ and B$'$ may be found when the plane has been turned about its vertical trace, as shown in the figure.

Conversely. If A be given to find a and a', draw A a perpen-

dicular to M N, and cutting it in a; draw A a'' perpendicular to $x\,y$, and meeting it in a''; with N as a centre, and a radius N a'', describe the circle $a''\,a'$, cutting N P in a'; a and a' will be the projections of A.

Fig. 6.



II. Let the point (a, a') (Fig. 7) be situated in the plane M N P, which is perpendicular to the horizontal plane only. N P will be perpendicular to $x\,y$ (*Euc.* XI. 19).

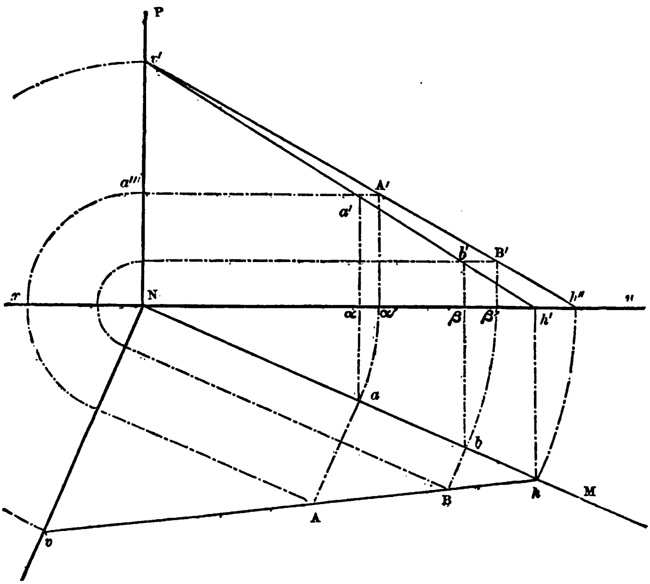
If the plane be turned about its horizontal trace M N, the distance of the point A from M N will evidently be equal to $a\,a'$, $a\,a'$ being a straight line perpendicular to $x\,y$ (30).

Draw a A perpendicular to M N, make it equal to $a\,a'$; A will be the position of the point (a, a') after the plane has been turned down. In the same manner B (b, b') may be found;

and thus the straight line A B passing through these points is determined.

Conversely. If A be given to find a and a'.

Fig. 7.



Draw A a perpendicular to M N; a will be the plan of A; its elevation will be in $a\,a'$, drawn from a perpendicular to $x\,y$ (30), and at a distance, $a\,a'$ from $x\,y$, equal to A a.

Again; if the plane be turned about its vertical trace. The point (a, a') will be situated in a straight line drawn through a' perpendicular to N P; the distance of A′ from N P will, by the principle enunciated (36), be equal to N a, the hypothenuse of the right angled triangle N $a\,\alpha$; because, in this case, the profile plane is parallel to the horizontal plane. In the same way B′ may be found, and the straight line A′ B′ determined. The point v', in which A′ B′ meets N P, will be the vertical trace of the line

$(a\,b,\ a'\,b')$: the point h'', in which $A'\,B'$ meets $x\,y$, will be the position which the horizontal trace of the line takes after the plane has been turned into coincidence with the vertical plane. If the constructions have been correctly performed the circle described with N as a centre, and radius $N\,h$, will pass through h''; h being the point in which $h'\,h$ drawn from h' perpendicular to $x\,y$ cuts M N; h' being the point in which $a'\,b'$ produced meets $x\,y$.

Conversely. To determine a and a' when A' is given. Draw $A'\,a'''$ perpendicular to N P; a' will be in this line; draw $A'\,a'$ perpendicular to $x\,y$; with centre N, and radius $N\,a'$, describe a circle cutting M N in a, a will be the plan of A; through a draw $a\,a'$ perpendicular to $x\,y$, and cutting $A'\,a'''$ in a'; a' will be the elevation of A.

III. Let the point (a, a') (Pl. I. Fig. 1) be in the plane M N P, which makes oblique angles with both of the planes of projection, and is not parallel to the ground line.

Referring to the principle set forth (36), the point (a, a') will, after the plane has been turned about its horizontal trace, be situated in the straight line drawn from a perpendicular to M N, its distance from M N being equal to the hypothenuse of a right angled triangle, whose base is $a\,a''$, and perpendicular $a\,a'$. This triangle may be constructed on the horizontal plane, as $a\,a''\,n$: or on the vertical plane, as $a\,a'\,m$. With centre a'', and radius $a''\,n$, describe a circle cutting $a\,a''$ produced in A. A will be the point required. A second point B may be found in the same manner, and thus the straight line A B determined.

Cor. It is evident that each of the angles $a'\,m\,a$ and $a\,a''\,n$ measures the inclination of the plane M N P to the horizontal plane.

As a particular case of this problem let it be required to construct the vertical trace N P. Find, as above, the position V of (v, v') a point in the vertical trace; join N V; N V will be the line required, and the angle M N V will be the angle contained by the traces in space.

Conversely. Let a and a' be required when A is given. The plane may be given either by its traces M N and N P; or by its

vertical trace N P, in the position N P′, and its horizontal trace M N. In both cases the right angled triangle $t\,v\,r$, corresponding to the construction of V (v, v'), may be described. For, if M N and N P be given, $v\,r = v\,v'$; and $t\,v$ are known. If M N and N P′ be given, $t\,v$ and $t\,r = t\,\mathrm{V}$ are known.

Draw A a'' perpendicular to M N; and construct the right angled triangle $a''\,a\,n$, in which $a''\,n = a''\,\mathrm{A}$; and the angle $a\,a''\,n =$ the angle $v\,t\,r$: a will be the plan of A; its elevation will be in $a\,a'$, perpendicular to $x\,y$, at a distance from $x\,y$, $a\,a' = a\,n$.

CHAP. II.

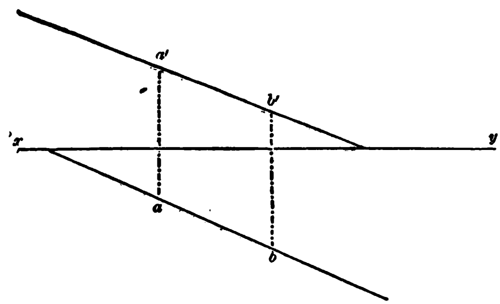
ELEMENTARY PROBLEMS ON STRAIGHT LINES AND PLANES.

Problem I.

Given the projections of two points, to find the projections of the straight line passing through the points.

Let $a\,a'$ and $b\,b'$ (Fig. 8) be the projections of the points.

Fig. 8.



Then, since the projections of the straight line A B must pass through the projections of the points, the line $a'\,b'$ will be the elevation, and $a\,b$ the plan of the line required.

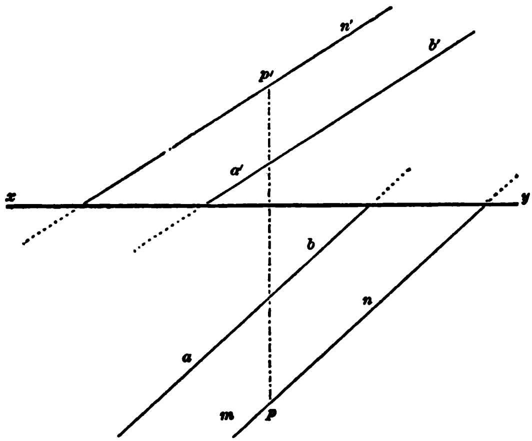
Problem II.

Through a given point, to draw a straight line parallel to a given straight line.

Let (p, p') (Fig. 9) be the given point, $(a\,b, a'\,b')$ the given line. Since the required line passes through the point (p, p'), its plan

and elevation will pass through p and p' respectively; moreover the projections of parallel straight lines are parallel (11). The

Fig. 9.



elevation of the line will therefore be $m'n'$ drawn through p' parallel to $a'b'$; its plan will be mn drawn through a parallel to ab.

Problem III.

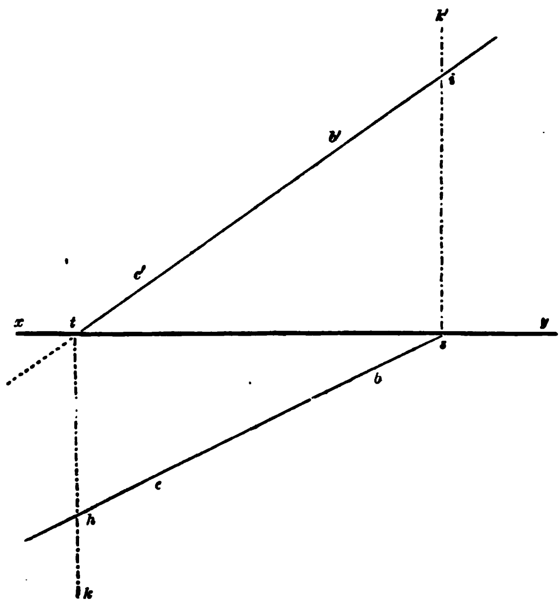
Given the projections of a straight line, to find its traces.

Let the given projections bc and $b'c'$ (Fig. 10) meet xy in s and t respectively. The elevation of the horizontal trace will be in xy (26); it will also be in the elevation $b'c'$, and must therefore be the point t. The trace itself will be in tk drawn from t perpendicular to xy in the horizontal plane, and also in the plan bc; it must consequently be the point h, in which bc and tk intersect. In the same manner it may be shown that, if sk' be drawn from s perpendicular to xy, and cutting $b'c'$ in i, i will be the vertical trace.

Generally. To find the horizontal trace of a straight line.

From the point in which the elevation meets $x y$, draw in the horizontal plane a perpendicular to $x y$; the point in which this perpendicular meets the plan of the line will be the horizontal trace.

Fig. 10.



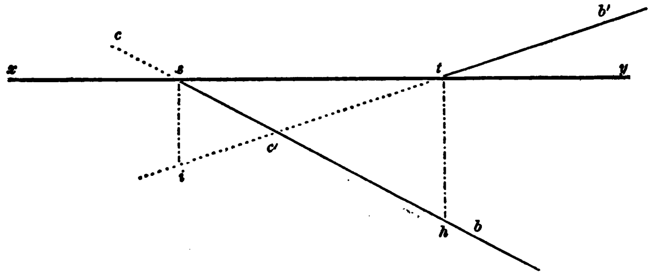
Similarly. From the point in which the plan meets $x y$, draw in the vertical plane a perpendicular to $x y$; the point in which this perpendicular cuts the elevation will be the vertical trace.

Obs. The portion of the straight line included between its traces may evidently have any one of the following positions, with reference to the planes of projection:—

(*a.*) It may be above the horizontal, and in front of the vertical plane (Fig. 10).

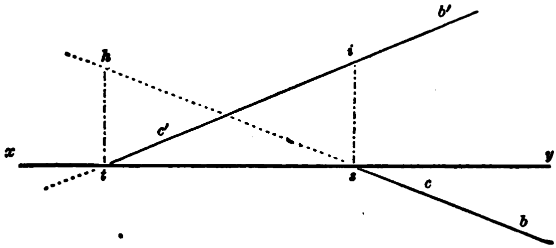
(*b.*) It may be below the horizontal, and in front of the vertical plane (Fig. 11).

Fig. 11.



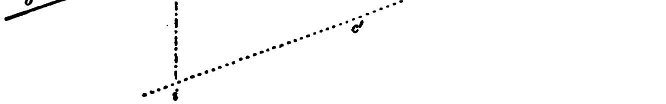
(*c*.) It may be above the horizontal, and behind the vertical plane (Fig. 12).

Fig. 12.



(*d*.) It may be below the horizontal, and behind the vertical plane (Fig. 13).

Fig. 13.



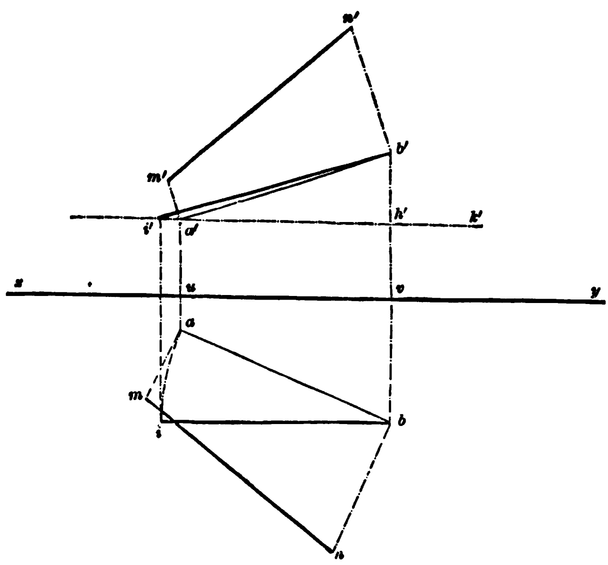
Cor. The construction of the converse of this Problem is obvious.

PROBLEM IV.

Given the projections of two points, to determine the length of the straight line joining the points.

Let (a, a') and (b, b') (Fig. 14) be the given points. The straight lines $a\,a'$ and $b\,b'$ will be perpendicular to $x\,y$ (29); let them cut $x\,y$ in u and v respectively. If vertical straight lines equal

Fig. 14.



to $u\,a'$ and $v\,b'$ be conceived to be drawn from a and b respectively, the extremities of these lines will be the points A and B in space, the distance between which has to be determined. Suppose a straight line drawn through the point A parallel to $a\,b$, and terminated by the vertical b B. The result will be a right-angled triangle, the base of which is equal and parallel to $a\,b$; the perpendicular, the difference between a A and b B, or, which

is the same thing, the difference between $u\,a'$ and $v\,b'$; the hypothenuse being the line sought to be determined.

If therefore through a', $h'\,i'$ be drawn parallel to $x\,y$, and equal to $a\,b$, and $b'\,i'$ be joined, $b'\,i'$ will be the line required. A similar construction made in the horizontal plane will give the same result.

Otherwise. The verticals a A and b B form with $a\,b$ and A B a trapezoid, whose plane is vertical. Imagine this trapezoid turned about b B until its plane is parallel to the vertical plane of projection. The base $b\,a$ will remain in the horizontal plane, but take the position $b\,i$, parallel to xy; the line A B, in its new position, will be parallel to the vertical plane, and will consequently be projected thereon in its real magnitude (13). The point B will remain fixed; its elevation will therefore still be b'. The point A will change its position, but not its distance from the horizontal plane; its elevation will thus be in the straight line $a'\,k'$ drawn through a' parallel to $x\,y$. Now the plan of A in its new position is i; if then a straight line be drawn from i perpendicular to $x\,y$, the point i' in which $i\,i'$ cuts $a'\,k'$ will be the elevation of A when the plane a A B b is parallel to the vertical plane of projection; and the straight line $i'\,b'$ will be equal to A B.

Or, A B may be determined by turning the trapezoid a A B b about $a\,b$ until it coincides with the horizontal plane; as $a\,b\,m\,n$: or, by turning the trapezoid a'A B b' about $a'\,b'$ until it coincides with the vertical plane as $a'\,b'\,m'\,n'$.

Then, $i'\,h'=a\,b$; $a\,m=u\,a'$; $a'\,m'=u\,a$; $b\,n=v\,b'$; $b'\,n'=v\,b$.

N.B.—This problem has been discussed at length, because it is one that frequently occurs.

Problem V.

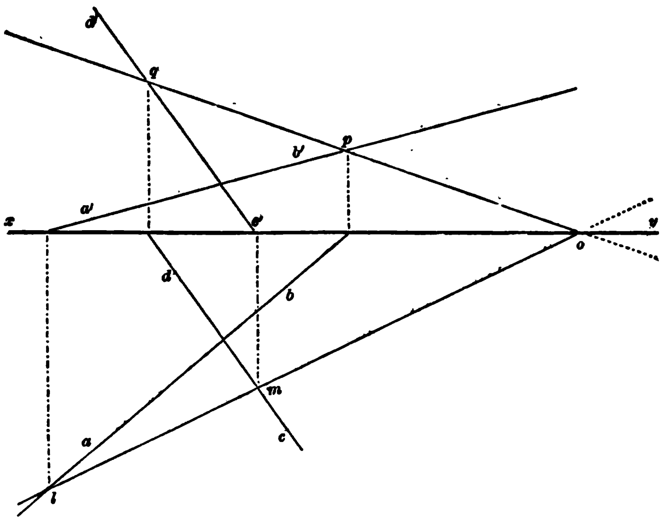
To draw a plane through two given straight lines that intersect, or are parallel.

The vertical traces of the lines will be two points in the vertical trace of the plane; the horizontal traces of the lines will be

two points in the horizontal trace of the plane. The traces are therefore determined thus :—

Let ($a\,b$, $a'\,b'$) and ($c\,d$, $c'\,d'$) (Fig. 15) be the given lines. Find by Prob. III. l and m, their horizontal traces, and p and q, their

Fig. 15.



vertical traces. The straight lines passing through p and q, and l and m, respectively, will be the traces of the required plane, and should meet $x\,y$ in the same point o.

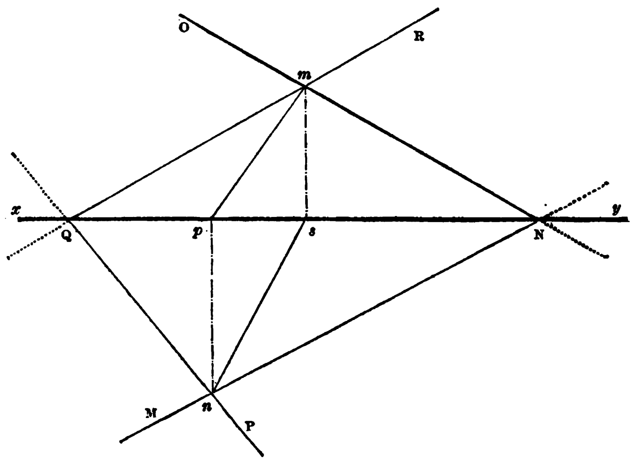
Problem VI.

To determine the projections of the intersection of two given planes.

Let M N O and P Q R (Fig. 16) be the given planes; then m_r the intersection of their vertical traces, will be the vertical trace of their intersection; and n will be its horizontal trace. The intersection itself is the straight line in space joining m and n. It is required to construct the projections of this line.

Draw *m s* perpendicular to *x y*; *s* will be the plan of *m* (26), and *n* is the plan of a second point in the line; consequently the

Fig. 16.



plan of the intersection is *s n* (6). In the same manner it may be shown that if *n r* be drawn perpendicular to *x y*, the elevation of the intersection is *p m*.

PARTICULAR CASES.

I. Let the horizontal traces M N and P Q be parallel (Fig. 17), then the intersection of the planes will be parallel to M N and to P Q; consequently, the plan of the intersection will also be parallel to M N and P Q (11), and its elevation will be parallel to *x y* (26). The point *m* in which the vertical traces of the planes cut each other, is a point in the intersection of the planes; if therefore *m s* be drawn perpendicular to *x y*, *m* and *s* will be the plan and elevation of *m*; *s d* parallel to M N, and *m p* parallel to *x y* will therefore be the projections required.

II. Let the traces of both planes be parallel to *x y*, as M N, M′ N′, and P Q, P′ Q′ (Fig. 18). The planes themselves will be parallel to *x y*, as will also their intersection.

A third plane of projection will now be necessary in order to determine the intersection. Let this plane be assumed perpen-

Fig. 17.

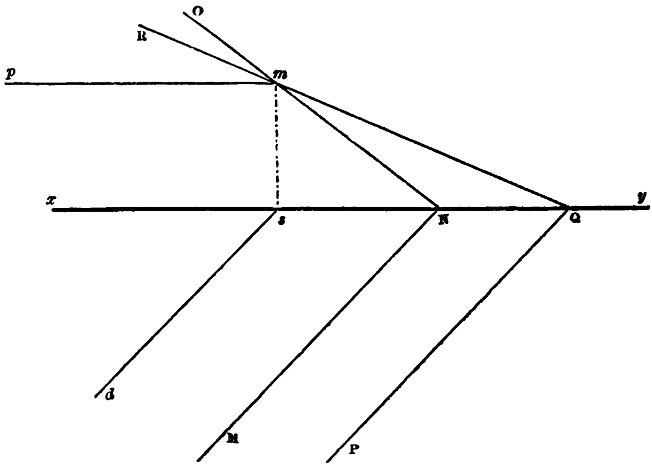
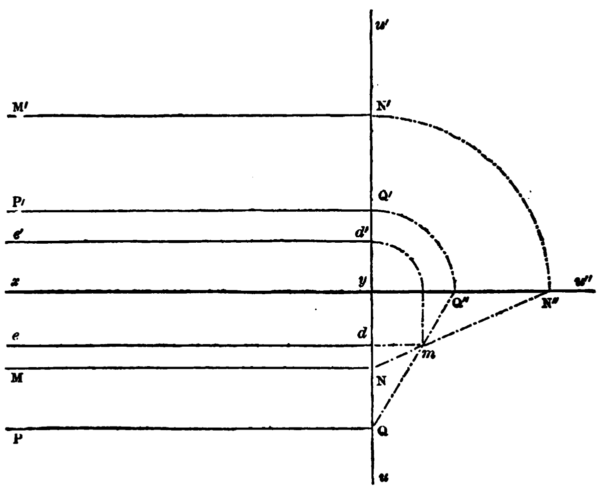


Fig. 18.



dicular to the other two, and therefore to $x\ y$ (*Euc.* XI. 19); its traces $y\ u$ and $y\ u'$ will be in one straight line perpendicular to $x\ y$ (27). Then N N″ and Q Q″, the traces of the given planes on the third plane of projection may be found by (35): the point m, in which these traces cut each other, will be the point in which the intersection of the planes meets the third plane; d and d' the projections of m, may be found by (35), and the straight lines $d\ e$, $d'\ e'$ drawn parallel to $x\ y$ will be the projections of the intersection.

III. Let the traces of both planes meet $x\ y$ in the same point: the intersection may be determined by a construction similar to that in case II.; and the solution is left as an exercise.

IV. If the traces of one plane be parallel to those of the other, but not parallel to $x\ y$; the planes are parallel, and therefore have no intersection.

Problem VII.

To determine the point of intersection of three given planes.

The planes combined two and two cut each other in three straight lines, which all pass through the required point. Let the projections of these intersections be constructed, by Prob. VI.; then if the constructions be accurately performed, the three elevations will pass through the point o' (Fig. 19); and the three plans will pass through o. The point (o, o') will be the point required: the straight line $o\ o'$ should be perpendicular to $x\ y$ (29).

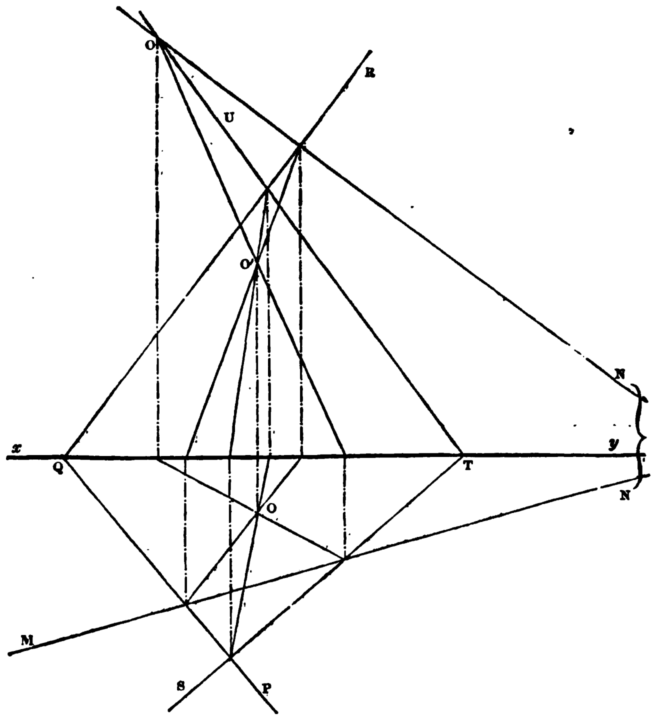
Problem VIII.

To draw a plane through three given points.

The three straight lines which join the given points, taken two and two, being situated in the required plane, will meet the planes of projection in points which belong to the traces of the plane. Thus, by Prob. III. may be found three points in each

trace. The three points in each trace must be in the same straight line; and the traces must meet $x\,y$ in the same point (21).

Fig. 19.



Let (a, a'), (b, b') and (c, c'), (Fig. 20) be the given points: $(a\,b, a'\,b')$, $(b\,c, b'\,c')$ and $(a\,c, a'\,c')$, the lines passing through those points taken two and two. Determine (Prob. III.), d, e, f, the horizontal, and d', e', f', the vertical traces of these lines. The straight line drawn through e and f should pass through d; and that drawn through e' and f' should pass through d': these straight lines $d\,e\,f$, $d'\,e'\,f'$, which are the required traces, must meet $x\,y$ in the same point t.

Obs.—The following cases may be considered with advantage:—

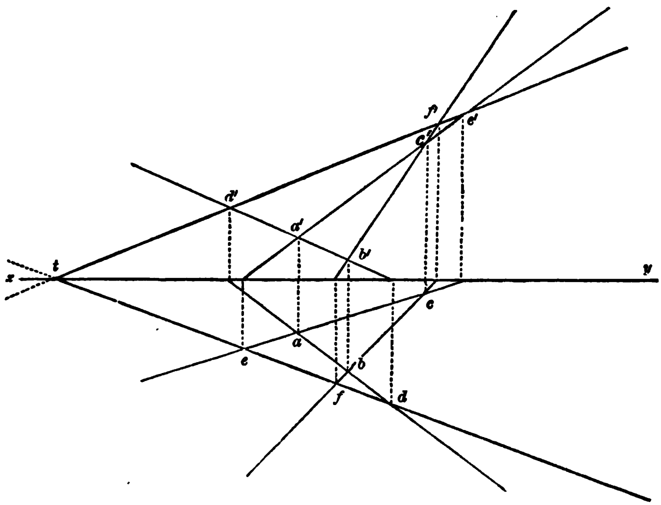
I. When one of the lines is parallel to one of the planes of projection.

II. When one of the lines is parallel to both planes of projection.

III. When two of the lines are parallel to one of the planes of projection.

IV. When the three given points are in a straight line.

Fig. 20.



PROBLEM IX.

To determine the point in which a given straight line meets a given plane.

If, through the plan of the given line, a vertical plane be drawn, this plane will contain the point sought; but the point is also in the given plane; it must therefore be that point in which the intersection of these two planes meets the given line.

Let $d\ n$ and $d'\ s'$ (Fig. 21) be the projections of the line; P Q R, the given plane: draw through $d\ n$ the vertical plane $d\ n\ n'$; its vertical trace $n\ n'$ will be perpendicular to $x\ y$ (26): draw $b\ b'$ perpendicular to $x\ y$: join $b'\ n'$: $b'\ n'$ will be the

elevation of the intersection of the planes P Q R and $d\ n\ n'$. (Prob. VI.) Now the elevation of the required point will be in $b'\ n'$, and also in $d'\ s'$; it must therefore be the point f' in which

Fig. 21.

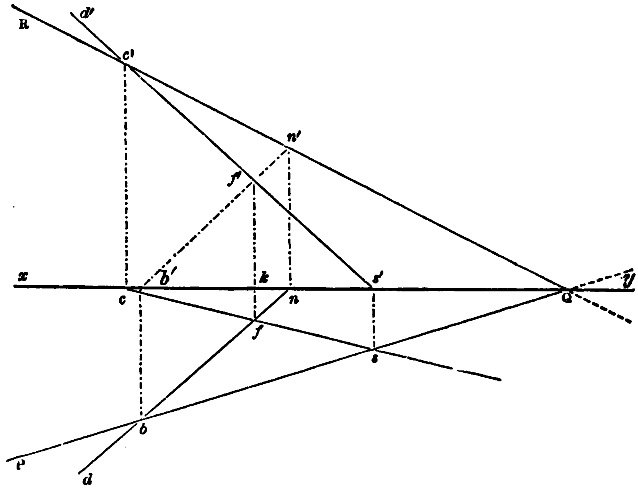
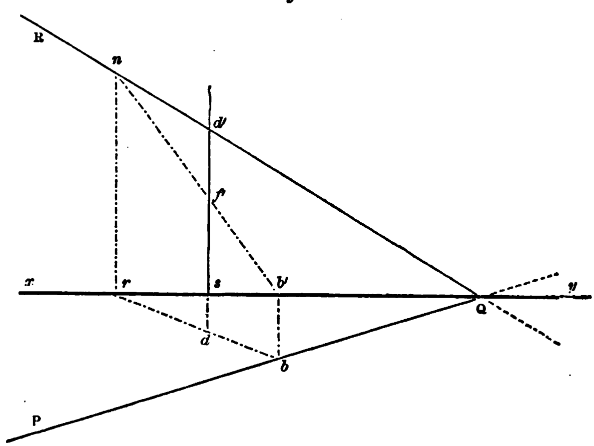


Fig. 22.



these straight lines cut each other. Through f' draw $f\,k$, perpendicular to $x\,y$, and cutting $d\,n$ in f: f will be the plan of the required point (29). The plan f might have been determined by an independent construction, similar to that employed for the

Fig. 23.

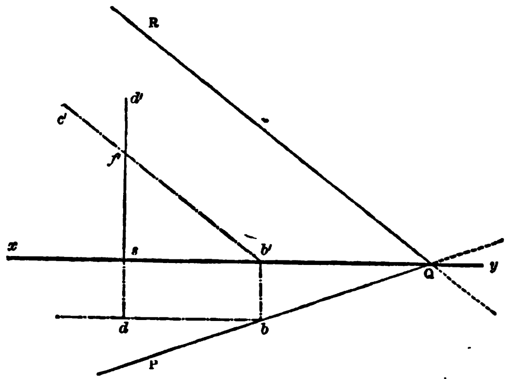
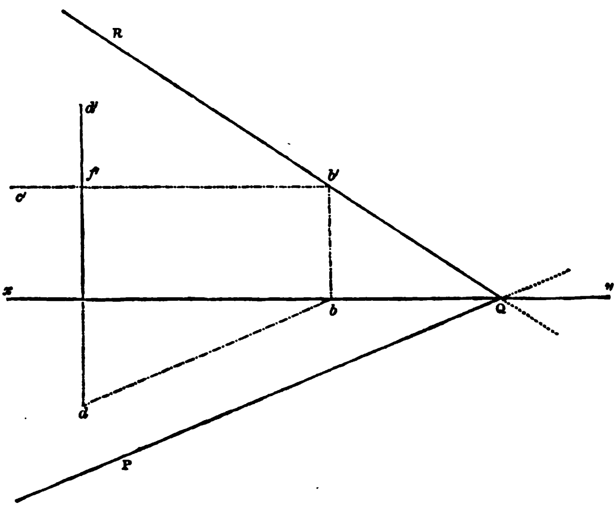


Fig. 24.



elevation, viz. by drawing through d' s' a plane perpendicular to the vertical plane, as shown in Fig. 21.

If the given line be vertical, its plan will be a point (10), as d, Fig. 22; which is also the plan of the required point; the elevation of the line will be d s d', perpendicular to $x y$ (26). To find the elevation of the point, draw through d a vertical plane, b r n; draw b b' perpendicular to x y; join b' n, cutting d d' in f' : f' will be the elevation required.

In this case, the vertical plane assumed, is subject only to the condition of passing through a given vertical straight line, it is consequently indeterminate; its trace may therefore be parallel to x y as in Fig. 23, or to the horizontal trace of the given plane, as in Fig. 24.

Problem X.

Through a given point to draw a plane parallel to a given plane.

Since every plane passing through the given point cuts the given plane and the plane required in two parallel straight lines (*Euc.* XI. 16); if through the given point a straight line be drawn parallel to any straight line in the given plane, this line will be in the required plane. The traces of this line will be points in the traces of the required plane. The traces of this plane will therefore (*Euc.* XI. 16) be the two straight lines drawn through these points, and parallel respectively to the traces of the given plane.

Let (d, d') (Fig. 25) be the given point, P Q R the given plane, take any point, m, in the horizontal trace P Q; and any point n', in Q R; draw n' n and m m' perpendicular to x y; join n' m' and m n, these lines will be the projections of a straight line $(m\ n, m'\ n')$ situated in the plane P Q R; because m and n' are the traces of a line in that plane (Prob. III.). Through (d, d') draw by Prob. II. a straight line parallel to $(m\ n, m'\ n')$, through f and g, the traces of this line (Prob. III.) draw M N and N O parallel to P Q and Q R respectively: M N O will be the plane required.

Solution 2. Let P Q R (Fig. 26) be the given plane: through the given point (d, d'), conceive a straight line H to be drawn parallel to P Q: this line will be in the plane required, and

Fig. 25.

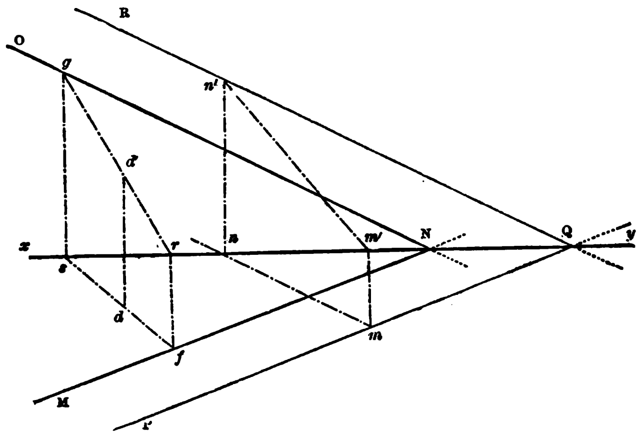
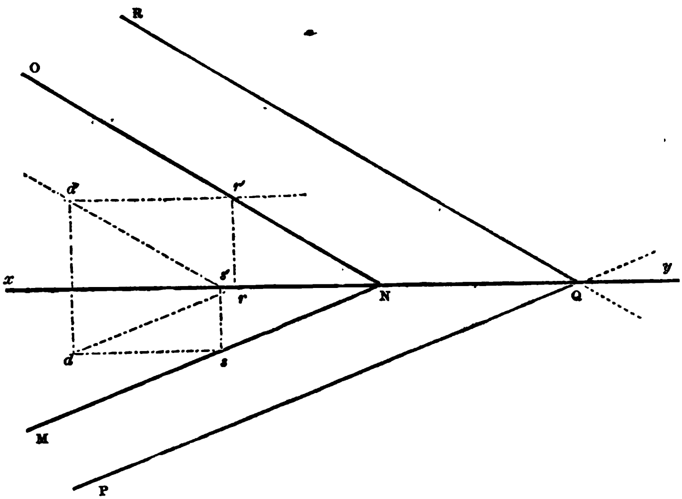


Fig. 26.



will be parallel to the horizontal plane. Its plan will be the line $d\ r$, drawn through d parallel to P Q (11): its elevation will be $d'\ r'$ parallel to $x\ y$ (21). The point r', in which H meets the vertical plane (Prob. III.), will be a point in the vertical trace of the plane required: the traces of which will therefore be O N drawn through r' parallel to R Q, and M N through N parallel to P Q (*Euc.* XI. 16).

In a similar manner, the problem may be solved by drawing through (d, d') a straight line parallel to the vertical plane, as shown in the figure.

If the given plane be parallel to one of the planes of projection as, for example, the vertical plane, it will have then only a horizontal trace which will be parallel to $x\ y$ (21). The required plane will also have only a horizontal trace which will be the straight line drawn through the plan of the given point parallel to $x\ y$.

Problem XI.

Through a given point to draw a straight line perpendicular to a given plane: to find the point in which the perpendicular meets the plane; and to determine the length of the perpendicular.

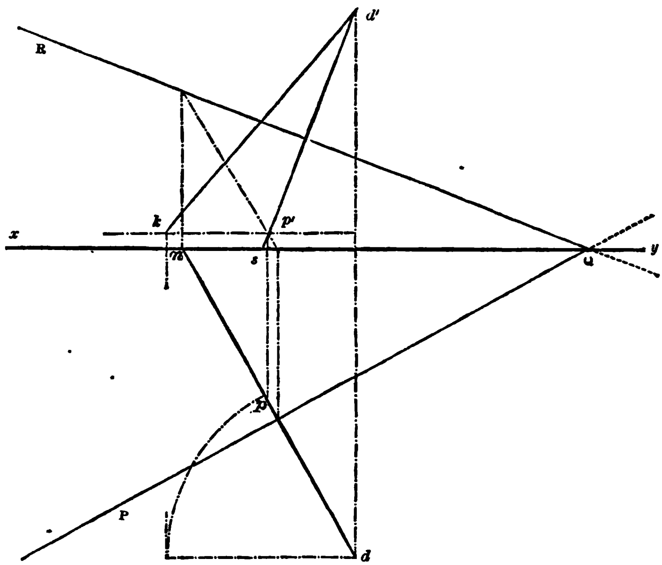
Let (d, d') (Fig. 27) be the given point, P Q R the given plane. The projections of the perpendicular must pass through d and d' respectively: they will also be perpendicular to the traces of the plane (30). If, therefore, through d and d', $d\ n$ and $d'\ s$ be drawn perpendicular to P Q and Q R, $d\ n$ and $d'\ s$ will be the projections required. The point (p, p') in which the perpendicular meets the given plane may be determined by Prob. IX., its length, $d'\ k$, may be found by Prob. IV.

Problem XII.

Through a given point to draw a plane perpendicular to a given straight line.

Let (d, d') (Fig. 28) be the given point, $(a\ b, a'\ b')$ the given line. The traces of the required plane will be perpendicular respectively to the projections of the given line (30). Consequently, the directions of the traces of the required plane, and

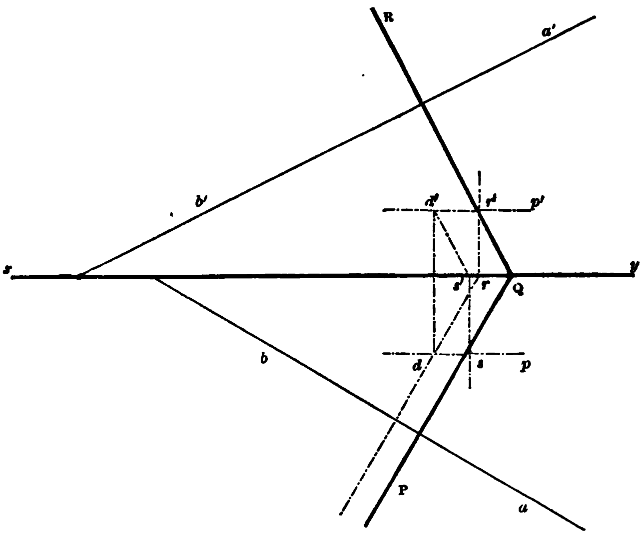
Fig. 27.



one point in that plane, are known; it will therefore be sufficient if a point in one of the traces be determined. To effect this, conceive a straight line to be drawn through the given point parallel to the horizontal trace of the required plane; this line will be situated in that plane, and will be parallel to the horizontal plane; its plan will therefore be parallel to the horizontal trace of the plane (11), and, consequently, perpendicular to the plan of the given line. But, since this line passes through (d, d'), its plan must be $d\ r$ perpendicular to $a\ b$; its elevation $d'\ p'$ parallel to $x\ y$ (*Euc.* XI. 16). Since $d\ r$ and $d'\ p'$ are the projections of a straight line in the required plane; if r', the vertical trace of this line, be determined by Prob. III., r' will be

a point in the vertical trace of the required plane; and the traces will be R Q drawn through r' perpendicular to $a' b'$; and Q P drawn through Q perpendicular to $a\ b$.

Fig. 28.



A verification may be obtained by drawing through (d, d') a straight line parallel to the vertical plane, and making a construction similar to the preceding one, as shown in Fig. 28.

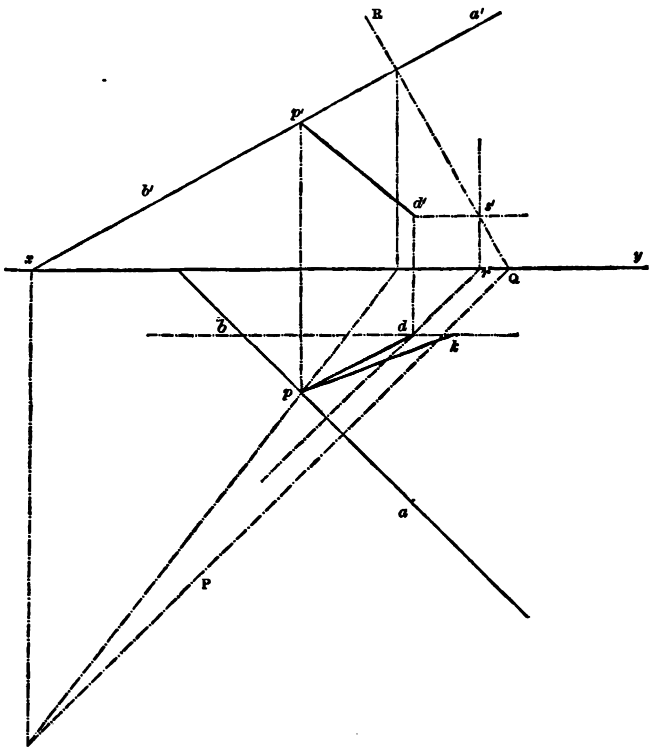
Problem XIII.

From a given point to draw a straight line perpendicular to a given straight line, and to determine the length of the perpendicular.

Let (d, d') (Fig. 29) be the given point, $(a\ b, a'\ b')$ the given line. Through (d, d') draw a plane P Q R perpendicular to $(a\ b, a'\ b')$ (Prob. XII.); determine (Prob. IX.) the point (p, p')

in which the plane P Q R meets the given line; then ($d\,p$, $d'p'$) will be the perpendicular required; its length, $p\,k$, may be determined by Prob. IV.

Fig. 29.



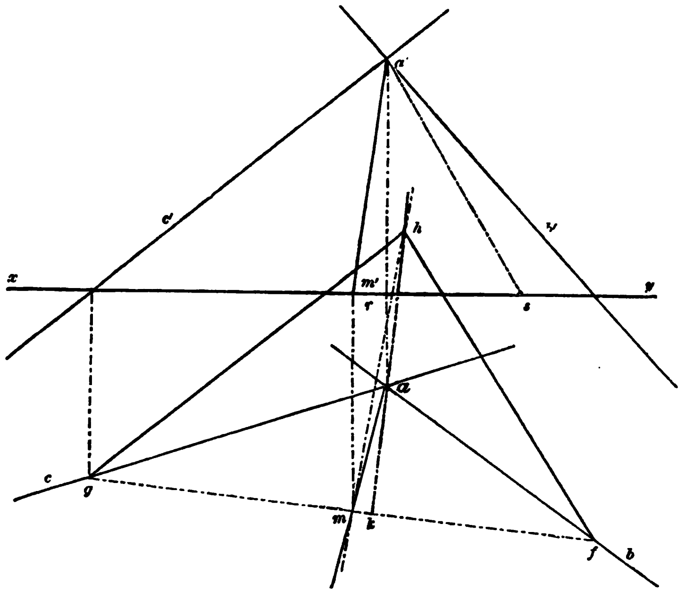
PROBLEM XIV.

Given the projections of two straight lines which cut each other, to determine the angle contained by the lines.

Let ($a\,b$, $a'\,b'$) and ($a\,c$, $a'\,c'$) (Fig. 30) be the given lines. Determine their horizontal traces g and f (Prob. III.); join $g\,f$;

g f and the portions of the straight lines intercepted between their point of intersection and their horizontal traces, form a triangle in which the angle opposite to *g f* is the angle sought to be determined. To construct this triangle, through *a* draw *h k* perpendicular to *g f*; conceive the plane of this triangle to be turned

Fig. 30.



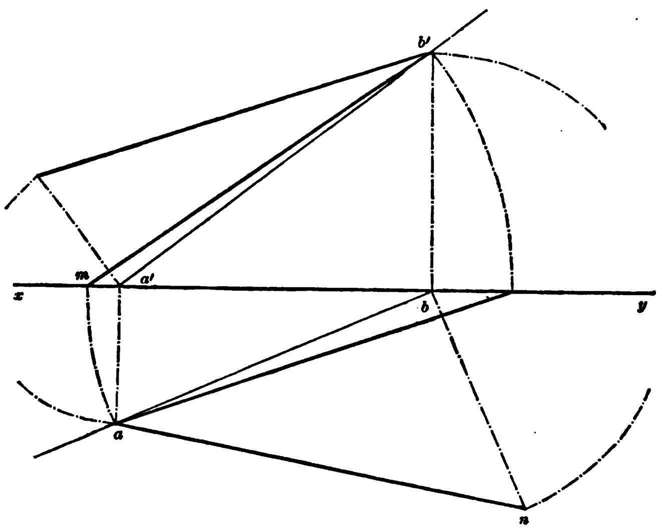
about its horizontal trace *g f*, until it coincides with the horizontal plane: *h*, the position which the point of intersection of the lines now takes, is found by (35); thus join *a a′*, let it cut *x y* in *r*; make *r s* equal to *a k*; join *a′ s*, make *h k* equal to *a′ s*; join *g h* and *f h*; *g h f* will evidently be the angle contained by the lines.

Problem XV.

Given the projections of a straight line to determine the angles which it makes with the planes of projection.

Let (*a b*, *a′ b′*) (Fig. 31) be the given line; find its traces *a* and *b′* (Prob. III.). Then the angle which the given line makes with the horizontal plane is the angle contained by the straight line in space drawn from *a* to *b′*, and the plan *a b* (*Euc.* XI. *def.* 5). If,

Fig. 31.



therefore, the plane *a b b′* be turned about *a b* until it takes the position *a b n* in the horizontal plane; *b n* being equal to *b b′*; *b a n* will be the angle required. The angle may also be determined by turning the plane *a b b′* about *b b′* until it takes the position *b′ b m* in the vertical plane, when *b′ m b* will be the required angle. The inclination of the line to the vertical plane may be found by a similar construction, as shown in Fig. 31.

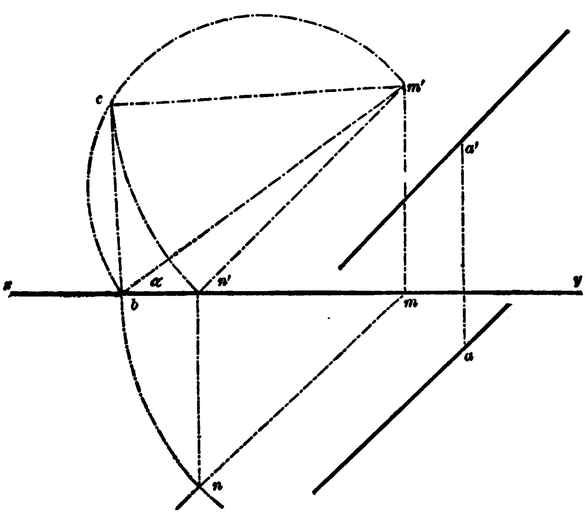
Conversely. Through a given point to draw a straight line, making given angles with the planes of projection.

This problem will be most easily solved by assuming a point on the vertical plane, and drawing through it a line having the given inclinations, and then drawing through the given point a line parallel to this line.

Let α and β be the angles which the line is to make with the horizontal and vertical planes respectively, it is evident that these are the angles between it and its projections.

Assume any point (m, m') (Fig. 31*a*) on the vertical plane; make the angle $m' b m$ equal to α; with centre m, and radius $m b$, de-

Fig. 31 *a.*



scribe the circle $b n$; all lines passing through (m, m') and making an angle α with the horizontal plane, will have their horizontal traces in the circumference of this circle (33, *Cor.* II.). Again, on $m' b$ describe a right angled triangle, having the angle $b m' c$ equal to β; make $m' n'$ equal to $m' c$ (the construction is shown in the figure), draw $n' n$ perpendicular to $x y$; n will be the

horizontal trace of the line. Its projections will evidently be $m' n'$ and $m n$.

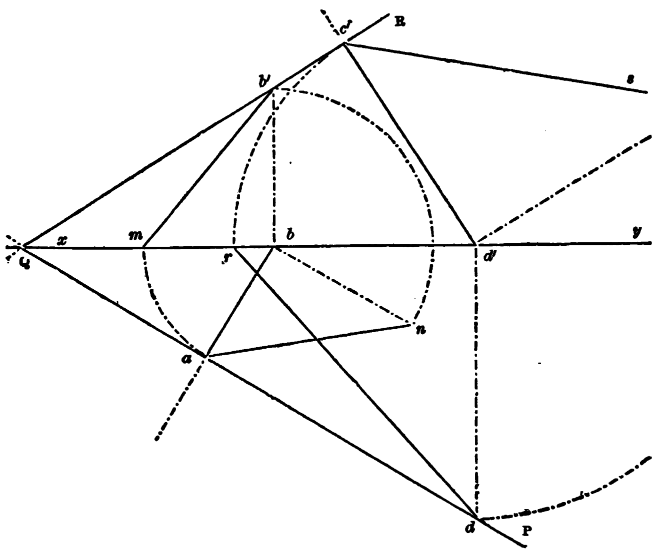
Let (a, a') be the given point; draw through a and a' parallels to $m n$ and $m' n'$ respectively, these will be the projections of the line required (11).

Problem XVI.

Given the traces of a plane, to determine the angles which it makes with the planes of projection.

Let P Q R (Fig. 32) be the given plane. To determine the angle which it makes with the horizontal plane, take any point a

Fig. 32.



in the horizontal trace P Q; through a draw a plane $a b b'$ perpendicular to the given plane and the horizontal plane; the profile angle of the given plane and the horizontal plane will

evidently be the angle required. The projections of the lines containing this angle are known, and thus the angle $b' m b$ may be determined by Problem XV. as shown in Fig. 32.

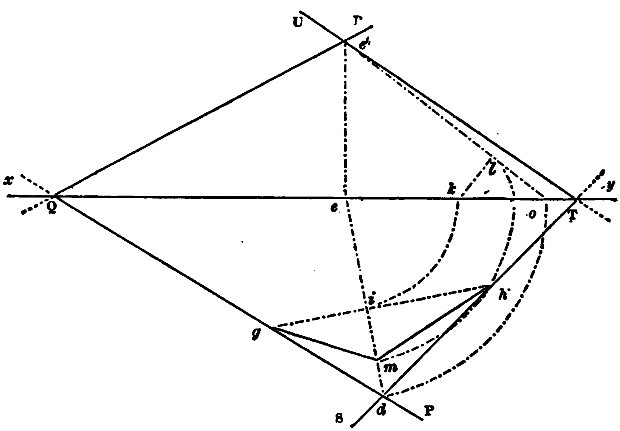
In a similar manner the inclination of the plane to the vertical, viz., the angle $d' r d$, may be found (Fig. 32).

Problem XVII.

To determine the angle contained by two given planes.

Let P Q R and S T U (Fig. 33) be the given planes. Find, by Prob. VI., $d e$ the plan of their intersection; let $g h$ be the horizontal trace of a profile plane, of the planes P Q R and S T U,

Fig. 33.



cutting $d e$ in i. Then, since this plane is perpendicular to the intersection of P Q R and S T U, $g h$ cuts $d e$ at right angles (30), it also forms, with the straight lines in which the profile plane cuts P Q R and S T U, a triangle, in which the angle opposite to $g h$ is the angle sought to be determined.

Conceive the plane of this triangle to be turned about $g h$ until it coincides with the horizontal plane; the vertex of the triangle

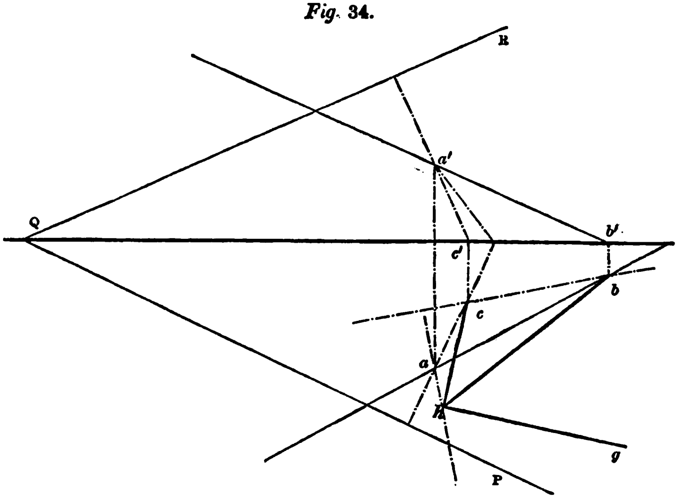
will then be in $e\,d$ (35). It only remains, therefore, to determine the altitude of the triangle, that is, the straight line drawn from i perpendicular to the intersection of the planes. This perpendicular evidently lies in the vertical plane $d\,e\,e'$; turn this plane about its trace $e\,e'$ until it coincides with the vertical plane; make $e\,k$ equal to $e\,i$; draw $k\,l$ perpendicular to $e'\,o$; $k\,l$ will be the altitude of the triangle required. If, then, $i\,m$ be made equal to $k\,l$, and $g\,m$, $h\,m$, be drawn, the angle $g\,m\,h$ will be the angle contained by the planes. This angle may be determined by a similar construction in the vertical plane.

Problem XVIII.

To determine the angle contained by a given straight line and a given plane.

Let P Q R (Fig. 34) be the given plane, ($a\,b$, $a'\,b'$) the given line. From any point (a, a') in the line, draw a straight line

Fig. 34.



perpendicular to the given plane (Prob. XI.); the projections of this perpendicular will be $a\,c$ and $a'\,c'$, perpendicular respectively

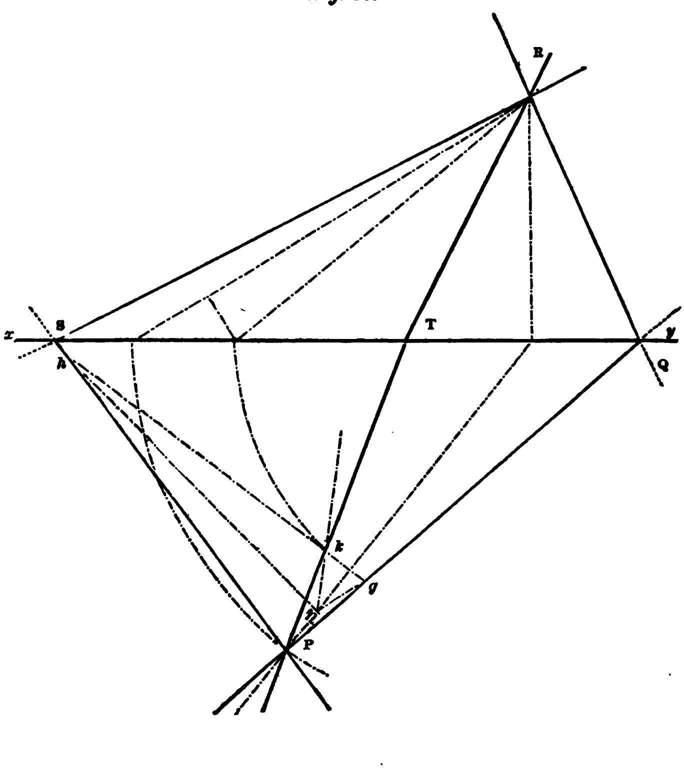
to PQ and QR the traces of the plane (30). The angle contained by this perpendicular and the given line will be the complement of the required angle; this angle between the perpendicular and the given line is found by Problem XIV. to be *b h c*; if, therefore, *h g* be drawn perpendicular to *h c*, *b h g* will be the angle required.

PROBLEM XIX.

To draw a plane bisecting the angle between two given planes.

Let P Q R and P S R (Fig. 35) be the given planes; *h n g* the angle between them, determined by Problem XVII. Bisect the

Fig. 35.



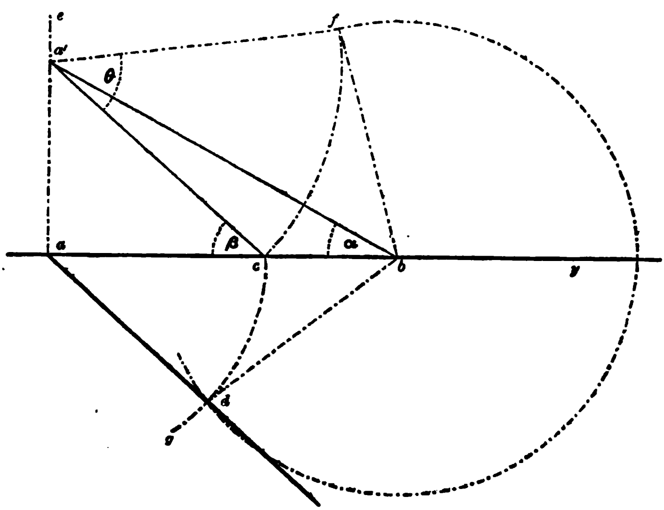
angle $h n g$ by the line $n k$, meeting $h g$ in k; k will be a point in the horizontal trace of the plane required. But P is also a point in that trace; the traces of the plane bisecting the angle will therefore be P T passing through k, and T R passing through R, since R is a point in the vertical trace.

Problem XX.

To reduce an angle to the horizon. That is, having given the angle which two straight lines make with each other, and their inclinations to the horizontal plane, to determine the plan of the angle.

Let a (Fig. 36) be the plan of the vertex of the angle: $a\,b$ that of one of the lines containing it, and inclined at an angle α to the horizontal plane; also, let the inclination of the

Fig. 36.



second line to the horizontal plane be β; θ being the angle contained by the lines.

Assume the vertical plane of projection to pass through the first line, so that $a\ b$ coincides with $x\ y$: from a draw $a\ e$ perpendicular to $a\ b$; the angular point will be in $a\ e$: let it be a': through a' draw $a'\ b$, making an angle α with $a\ b$: b will be the horizontal trace of the first line. Through a' draw $a'\ c$, making an angle β with $a\ b$: with a as a centre and radius $a\ c$ describe the circle $c\ d\ g$: all straight lines passing through a' and making an angle β with the horizontal plane, will have their horizontal traces in the circumference of this circle (33). Again, make the angle $b\ a'\ f$ equal to θ; and $a'\ f$ equal to $a'\ c$: join $b\ f$; with b as a centre and radius $b\ f$ describe a circle cutting the circle $c\ d\ g$ in d; draw $a\ d$: $a\ d$ will be the plan of the second line.

For the straight line in space joining a' and d, makes an angle β with the horizontal plane: it also makes an angle θ with $a'\ b$: for in the triangles $a'\ d\ b$ and $a'\ f\ b$, $d\ a'$ and $a'\ b$ are respectively equal to $f\ a'$ and $a'\ b$; the base $b\ d$ is equal to the base $b\ f$; therefore the angle $d\ a'\ b$ is equal to the angle $b\ a'\ f$ (*Euc.* I. 8), which is equal to θ (*cons.*) The line $a'\ d$ thus fulfils the conditions of the problem, and its plan is $a\ d$: therefore the angle $d\ a\ b$ is the plan of the angle required.

Problem XXI.

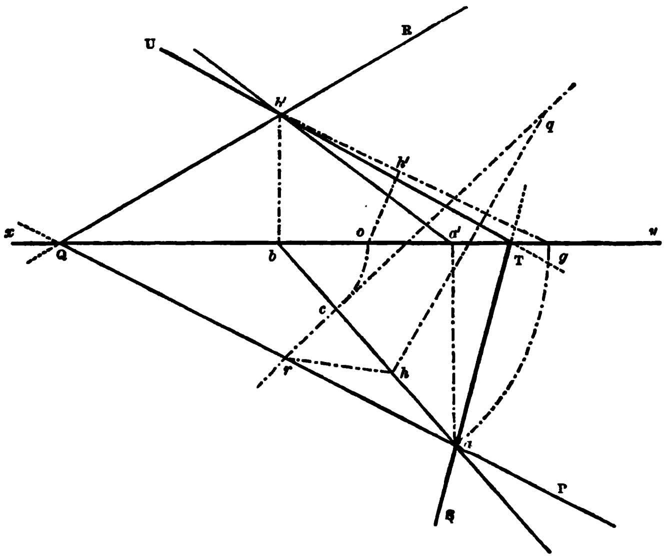
Through a straight line, in a plane, to draw a plane making a given angle with that plane.

The straight line will evidently be the intersection of the given and the required planes.

Let ($a\ b$, $a'\ b'$) (Fig. 37) be the given line, P Q R the given plane: draw a straight line $q\ r$ perpendicular to $a\ b$, and cutting it in c: $q\ r$ will be the horizontal trace of a profile plane of the given and required planes (30); and the lines in which this plane cuts the two planes will form the angle which these two planes make with each other (32). The vertex of this angle will be in the vertical plane passing through $a\ b$, and in the straight line drawn from the point c in $q\ r$ perpendicular to the line ($a\ b$, $a'\ b'$). Turn the plane $a\ b\ b'$ about $b\ b'$ until it coincides with

the vertical plane of projection: the line $(a\,b,\ a'\,b')$ and the point c, will take the positions $g\,b'$ and o respectively. Draw $o\,h'$ perpendicular to $g\,b'$; set off $c\,h$, in $c\,a$, equal to $o\,h'$: make the angle

Fig. 37.



$r\,h\,q$ equal to the given angle; q will be the horizontal trace of a line in the plane required, and consequently a point in the horizontal trace of the plane itself. The required traces will therefore be S a T q and U b' T: S T U being the plane required.

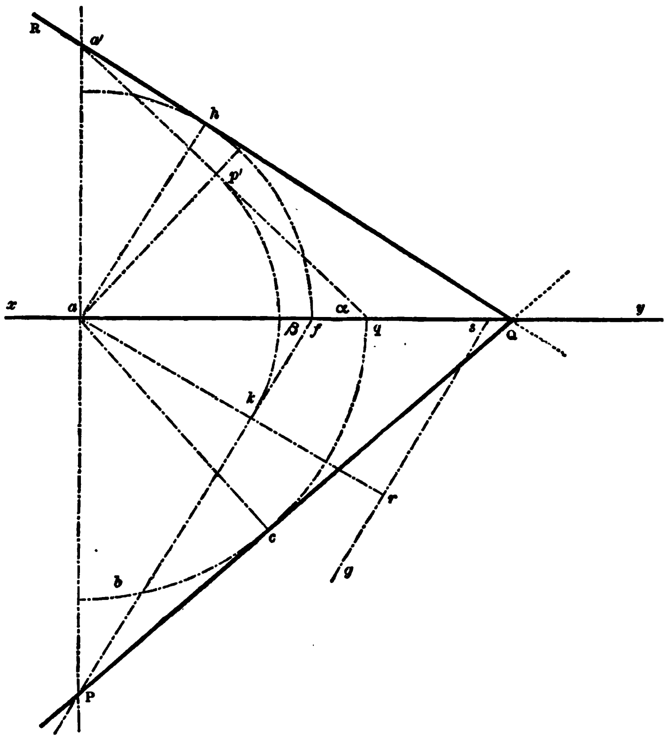
Problem XXII.

Through a given point to draw a plane making given angles with the planes of projection.

Let the given point $(a,\ a')$ (Fig. 38), be assumed in the vertical plane of projection; α and β being the angles which the required

plane is to make with the horizontal and vertical planes respectively. Make the angle $a' \, q \, a$ equal to α : with centre a and radius $a \, q$ describe a circle $q \, c \, b$: the horizontal trace of the plane will be a tangent to this circle. Draw the line $g \, s$, making with $x \, y$ the angle $g \, s \, x$ equal to β : draw $a \, p'$ perpendicular to

Fig. 38.



$a' \, q$; $a \, r$ perpendicular to $g \, s$, and make $a \, k$ equal to $a \, p'$: through k draw P k f, parallel to $g \, s$ and meeting $x \, y$ in f, $a' \, a$ produced, in P : with centre a and radius $a \, f$ describe the circle $f \, h$; the vertical trace of the required plane will be a tangent to this circle. Through P draw P Q, touching the circle $b \, c \, q$ in c :

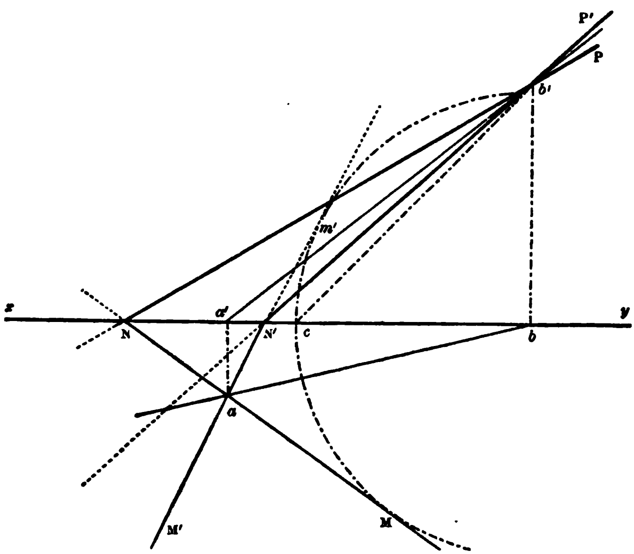
through Q draw Q R, touching the circle $f h$ in h. P Q R will be the plane required. This problem is indeterminate, since two planes can be drawn to fulfil the conditions.

Problem XXIII.

Through a given straight line to draw a plane, making a given angle with the horizontal plane.

Find, by Prob. III. the traces a and b' (Fig. 39) of the given line ($a\ b$, $a'\ b'$). These will be points in the traces of the required plane. It will therefore be sufficient if a second point in one of

Fig. 39.



the traces be determined. To effect this, in the horizontal trace, at b' in $b\ b'$ make the angle $b\ b'\ c$ equal to the complement of the given angle, with b as a centre and radius $b\ c$ describe the circle $c\ m$. If this circle cuts $a\ b$, the tangents a M and $a\ m'$, drawn

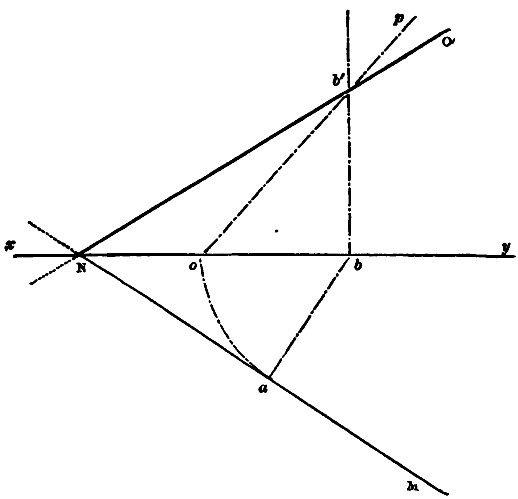
to it from a, will be the horizontal traces of two planes fulfilling the conditions of the problem. If the circle passes through a, only one such plane can be drawn: because the inclination of the plane will be equal to that of the given line. If a falls within the circle the problem is impossible, since the inclination of the plane would be less than that of the line (33). The planes, as determined in Fig. 39, will be M N P and M′ N′ P′ since the vertical traces must pass through b'.

Problem XXIV.

To find the vertical trace of a given plane, when its horizontal trace, and inclination are given.

Let M N (Fig. 40) be the given trace; α the angle at which the plane is inclined to the horizontal plane. Through any point

Fig. 40.



a in M N draw a vertical plane $a\ b\ b'$, perpendicular to M N; turn this plane about $b\ b'$ until it coincides with the vertical

plane of projection: the point a will take the position o; $b\,o$ being equal to $b\,a$: make the angle $b\,o\,p$ equal to α; b' the point in which $o\,p$ cuts $b\,b'$ will be a point in the vertical trace of the plane: through b' draw N O; this will be the vertical trace.

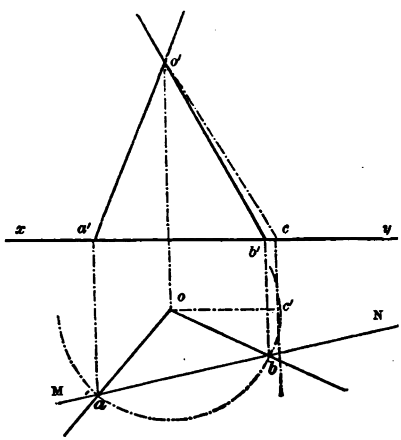
Should the trace M N not meet $x\,y$ within the limits of the drawing, a second point in the vertical trace must be determined in the same manner as b'.

Problem XXV.

Through a given point to draw a straight line which shall make a given angle with the horizontal plane, and meet a given straight line in that plane.

Let (o, o') (Fig. 41) be the given point, M N the given line situated in the horizontal plane. At o' in $o\,o'$, make the angle

Fig. 41.



$o\,o'c$ equal to the complement of the given angle; draw $c\,c'$ perpendicular to $x\,y$; and $o\,c'$ perpendicular to $c\,c'$. With o as a centre, and radius $o\,c'$, describe a circle cutting M N in a and b; draw $a\,a'$ and $b\,b'$ perpendicular to $x\,y$; join $o'\,a'$ and $o'\,b'$; the

two lines $(o\,a,\ o'\,a')$ and $(o\,b,\ o'\,b')$ will evidently fulfil the conditions of the problem (33 *Cor.* II).

This problem depending upon the intersection of a straight line and a circle, will have one solution, or two solutions, or will be impossible.

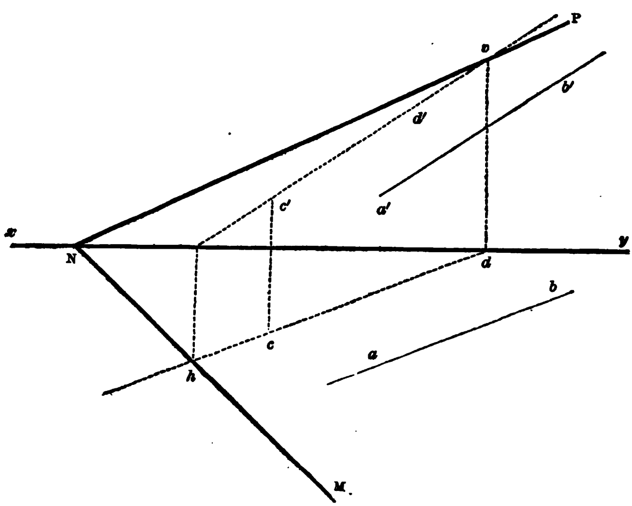
Problem XXVI.

Through a given point to draw a plane parallel to a given straight line.

Let $(c,\ c')$ (Fig. 42) be the given point, $(a\,b,\ a'\,b')$ the given line.

Through $(c,\ c')$ draw a straight line $(c\,d,\ c'\,d')$ parallel to $(a\,b,\ a'b')$ (Prob. II.); determine its traces h and v (Prob. III.). Any

Fig. 42.



plane, M N P, whose traces pass through h and v respectively, will be parallel to $(a\,b,\ a'\,b')$. The problem is therefore indeterminate.

Problem XXVII.

To draw a straight line bisecting the angle between two given straight lines.

The straight line bisecting the angle between the two lines is a line in the plane of these lines, and passing through their point of intersection. Determine the angle contained by the lines (Prob. XIV.); bisect it: the point in which the bisecting line meets the straight line joining the horizontal traces, will be a point in the plan of the bisecting line; this point and the projections of the angular point are sufficient to determine the projections required. The construction may be seen in Fig. 30, where ($a\,m$, $a'\,m'$) is the bisecting line.

Problem XXVIII.

Through a given point to draw a straight line parallel to a given plane.

This problem is indeterminate, since a straight line drawn parallel to any line in the plane will fulfil the condition (Prob. II.).

Problem XXIX.

Through a given point to draw a straight line parallel to two given planes.

Construct the intersection of the planes (Prob. VI.); and through the given point draw a straight line parallel to the intersection (Prob. II.). This will be the straight line required.

Problem XXX.

Through one given straight line to draw a plane parallel to another given straight line.

Through any point in the first line, draw a line parallel to the

second (Prob. II.); then by Prob. V. draw a plane containing this parallel and the first line. This will be the plane required. This problem is always possible; but it will be indeterminate when the two given lines are parallel, for in that case every plane passing through the first line, and not through the second, will be parallel to the second.

Problem XXXI.

Through a point in one given plane to draw a straight line parallel to another given plane.

Construct the intersection of the planes, Prob. VI., and through the given point, draw a straight line parallel to the intersection. This will be the line required.

Problem XXXII.

Through a given point to draw a straight line perpendicular to a given straight line.

Draw through the given point a plane perpendicular to the given line (Prob. XII.). Find the point in which the given line meets this plane (Prob. IX.). The straight line joining this point and the given point will be the perpendicular required. (*Euc.* XI. Def. 3.)

Problem XXXIII.

Through a given point to draw a plane perpendicular to two given planes.

Find the intersection of the given planes (Prob. VI.); draw through the given point a plane perpendicular to this intersection (Prob. XII.). This will be the plane required (*Euc.* XI. 18).

Problem XXXIV.

Through a given straight line to draw a plane perpendicular to a given plane.

From any point in the straight line, draw a straight line perpendicular to the given plane (Prob. XI.). The plane containing this perpendicular, and the given line, will be the plane required (Prob. V., *Euc.* XI. 18).

PROBLEM XXXV.

Given the plan of a polygon situated in a given plane; to find its elevation and real magnitude.

Let $a\,b\,c\,d\,e$ (Pl. I. Fig. 2) be the plan of a polygon, situated in the given plane P Q R. The elevations of the angular points of the polygon will be found by constructing the elevations of the points in which vertical lines drawn through a, b, c, d and e, respectively, meet the plane P Q R. These elevations may be constructed by Problem IX. Let them be a', b', c', d', e'. The figure $a'\,b'\,c'\,d'\,e'$ will be the elevation required. The angular points of the polygon A B C D E itself are found by turning the plane P Q R about the trace P Q in the manner explained in (35, III.).

PROBLEM XXXVI.

To determine the projections of a polygon situated in a given plane.

Let A B C D E (Pl. I. Fig. 2) be the given polygon; P Q R the given plane. The point a, which is the plan of A, is situated in A a, drawn from A perpendicular to P Q and at a distance from P Q equal to $o''a$, the base of the right-angled triangle $o''a\,a''$; in which triangle the angle $a''o''a$ is equal to the angle $n\,o''o'$, the inclination of the plane P Q R to the horizontal plane (Prob. XVI.); the hypothenuse $a''o''$ is equal to A m (35). In the same way b, c, d, and e may be found, and the plan completed by joining the respective points. Again, the side $a\,a''$ is the height of the point A above the horizontal plane. If, therefore, $a\,a'$ be drawn perpendicular to $x\,y$, and $a''a'$ parallel to $x\,y$, the point a', in which these lines meet, will be the elevation of A. Similarly, b', c', d', and e' may be found, and the elevation completed.

Problem XXXVII.

To determine the projections of a circle situated in a given plane.

1. Let the plane of the circle be parallel to one of the planes of projection, and perpendicular to the other. Its projection on the first will be a circle equal to the original one (13): its projection upon the second will be a straight line equal to the diameter of the circle (12 and 13).

2. Let the plane of the circle be inclined at any angle to the planes of projection.

Let A C D F H (Pl. I. Fig. 3) be the circle; P Q R the plane in which it is situated. Divide the circumference into equal parts in the points A, B, C, D, E, F, G, H: determine a, b, c, d, e, f, g, h, the plans, and a', b', c', d', e', f', h', the elevations of these points, as in Prob. XXXVI., by (35). The curves $a\,b\,c\,d\,e\,f\,g\,h$ and $a'b'c'd'e'f'g'h'$ will be the projections required.

These curves will be ellipses, whose major axes, being the projections of diameters parallel respectively to the planes of projection, are equal to the diameter of the circle.

Exercises.

1. Define the terms: trace of a line, trace of a plane, projecting line, projecting plane, sectional elevation, plane of projection.

2. Define the inclination of a straight line to a plane; define also a cone, a prism, a dihedral angle, and a trihedral angle.

3. If two straight lines be parallel, their plans are also parallel. Prove this, and point out in what case the plan of an angle is equal to the original angle, and whether it is otherwise greater or less.

4. The horizontal projection of a line is 6 ft. long, and inclined to the axis, or line of level, at an angle of 30°. The vertical

projection meets the axis at an angle of 20°. Construct the length of the line on a scale of $\frac{1}{20}$.

5. A plane is inclined 50° to the horizon; draw the plan of the locus of the intersection of all straight lines making an angle of 40° with the plane, proceeding from a point 2 inches distant from it.

6. The observed angle from a point A to two points B and C, of which the measured altitudes above the horizon, from the same point, are 30° and 35° respectively, is 45°. Construct the horizontal angle between the lines A B and A C.

7. Two planes intersect so that their horizontal traces are at right angles to each other, the line of intersection is inclined 30° to the horizon, one plane is inclined 55° to the horizon; what will be the inclination of the other plane?

8. Draw the plan of a regular pentagon of 1·5 inches side, its plane being inclined 49° to the horizon, and one of its sides horizontal.

9. A square of 2·3 inches side, the plane of which is inclined at 47°, has one side inclined at 26° to the horizon. Draw the plan of the square.

NOTE.—The Examples at the end of Chap. III. may be solved as additional exercises on this chapter.

CHAP. III.

HORIZONTAL PROJECTION.

1. Whatever be the surface under consideration, the elements necessary to determine it must always be found in its definition, in its generation, or in its properties. So that, these elements being known, the solutions of all problems relative to that surface may readily be deduced therefrom.

2. Every surface may be considered as having been generated by the movement of some line, constant or variable in magnitude.

3. *Def.*—The generating line is called the *generatrix* of the surface.

4. *Def.*—The movement of the generatrix is regulated by one or more fixed lines; these lines are called the *directrices* of the surface.

5. A surface may be represented graphically by the lines which constitute its generation; a mode of representation remarkable alike for elegance and simplicity, as will hereafter be seen. Since, however, each particular surface is capable of being generated in various ways, that method of representation should be chosen which is simplest, and best adapted to the problem under discussion.

6. In the preceding Chapter a plane has been represented by its traces on the planes of projection. It may be regarded as if generated by the movement of a straight line resting on both of the traces; or by a straight line which, resting on one trace, or on any straight line in the plane, moves parallel to the other

trace. In this case, should the generating line move parallel to the horizontal trace, the plane would be represented by a series of straight lines parallel to the horizontal trace. A plane may also be generated by a straight line turning perpendicularly about a fixed straight line which it always meets in the same point.

7. *A spherical surface* is generated by the revolution of a circle about one of its diameters which remains fixed. Here the generatrix is a circle, constant in magnitude; the fixed diameter is the directrix, and the surface may be represented as shown in Fig. 43.

The spherical surface may also be considered as generated by a circle, whose plane is perpendicular to a diameter of the sphere, moving so that its centre is always in that diameter; its radii, in its various positions, being the corresponding semi-chords of the great circle of the sphere. In this case the directrix is the diameter containing the centres, whilst the generatrix is a circle

Fig. 43.

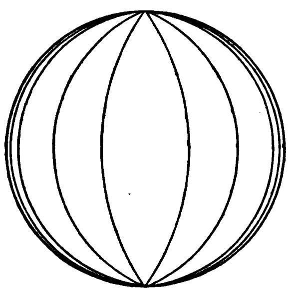
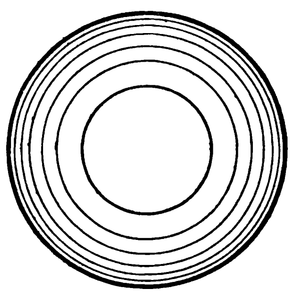


Fig. 44.



variable in magnitude. The surface may be represented as shown in Fig. 44. Since every diameter affords two kinds of generation, the surface is evidently susceptible of an infinite number of generations.

8. *A cylindrical surface* is generated by a straight line which, resting upon some curve, moves parallel to itself. From this method of generation it follows that the intersection, or intersec-

tions, of a cylindrical surface by a plane will be one or more straight lines.

A cylindrical surface may also be generated by a plane curve, which, moving parallel to itself, has one of its points resting on a fixed straight line.

9. *Def.*—If a cylindrical surface be cut by two planes perpendicular to the generating straight line, the portion intercepted between the planes is termed a *right cylinder;* and, if the directrix be a circle, the cylinder will be the same as that generated by the revolution of a rectangle about one of its sides.

10. *Def.*—*A conical surface* is generated by a straight line, which, passing through a fixed point, called *the vertex*, moves about that point in accordance with some law. It is, therefore, evident that the intersections of a conical surface by a plane passing through the vertex, will be two straight lines.

11. When the generatrix moves about the vertex in such a manner that a second point in it describes a circle, whose plane is perpendicular to the straight line passing through its centre and the vertex, the cone is called a right cone. Such a cone would be represented by a series of concentric circles, the distance between two consecutive ones being constant.

12. *Def.*—The *axis* of a cone is the straight line drawn through the vertex and the centre of the directrix.

13. *Def.*—The *sheets* of a cone are the surfaces separated by the vertex.

14. *Def.*—A curve surface may be considered as a polyhedron (Chap. IV.), whose faces are infinitely small; a *tangent plane* will then be nothing more than one of these faces indefinitely extended. Thus, to obtain a clear conception of a tangent plane to a surface in a point, imagine an indefinitely small portion of the surface around the point to be taken; this may be considered as lying in one plane, and that plane indefinitely extended, will be the tangent plane. Or thus: conceive any number of lines drawn through the point of contact on the surface, and take infinitely small portions of such lines around the point; these will

all be situated in the tangent plane. These small portions of the lines may be considered as straight lines, which, if produced indefinitely, will be tangents to lines on the surface; of each of these tangents an indefinitely small portion lies in the tangent plane, they must therefore lie wholly in that plane. The *tangent plane* to a surface in a point is therefore the plane containing the tangents, drawn through the point of contact, to all lines that can be drawn on the surface, through the point of contact.

15. It is manifest (14) that, if a plane touch a cylinder or a cone in a point, the generating straight line passing through that point lies entirely in the tangent plane; this property will be made available hereafter in drawing tangent planes. It is further evident that if the cone be a right cone with circular base, the tangent plane will be inclined to the base at the same angle as the generatrix; since the tangent plane will be perpendicular to the plane passing through the point of contact and the axis.

These brief remarks on the generation of surfaces have been inserted here as introductory to what follows on horizontal projection, in which branch of the subject it is often found convenient to represent a surface by means of the plans of its generatrix in various positions.

16. It has been shown in the preceding chapters that all points and lines in space may be represented by means of their projections on two planes cutting each other at right angles. In some cases, however, it is more suitable to represent objects by their plans only. This is especially the case in fortification and topographical drawings, in which, as many of the lines are nearly horizontal, the elevations would often intersect beyond the limits of the drawing. The elevations thus dispensed with are replaced by indices affixed to the plans of the various points, and denoting the vertical distances of those points from a given horizontal plane, called the *plane of reference*, or *comparison*.

17. *Def.*—This kind of projection is known as *horizontal projection*.

18. *Def.*—*A figured plan* is a plan which has attached to it a

number, showing, in units of the scale of the drawing, the vertical distance from the plane of reference of the point of which it is the plan. This number, as stated above, is termed the *index of the point.*

19. The indices of all points in the plane of reference will evidently be zero; whilst those of all points below that plane will be negative. Negative indices may however be avoided by assuming the plane of reference to have an elevation not greater than that of any point in the drawing. In the following problems the horizontal plane of projection will be assumed as the plane of reference. The horizontal trace of a line will then be that point in the line whose index is zero. This will be termed the zero point of the line. Again, since all points in a horizontal line, or in a horizontal plane, are equidistant from the plane of reference, they must evidently have the same index; and when such line is the horizontal trace of a plane, its index will be zero, and it will be simply called the trace of the plane.

20. As an illustration of the foregoing definitions it may be stated that the point lettered thus, $P_{,}p$, would denote a point in space whose plan is P, and whose vertical distance from the plane of reference is p units of the scale of the drawing. If the unit were ·1 inch, $P_{,}3$ would be a point ·3 in. above the plane; $P_{,}$-3 a point ·3 in. below it.

21. It is manifest that a point is known when given by its figured plan. For any straight line drawn in the plane of reference may be considered as the intersection of that plane by a vertical plane; and on this plane an elevation may be made in accordance with Chap. I.

This is what must be understood when in the following problems the expression "make an elevation of the point" occurs.

A straight line is determined by the figured plans of two of its points, since only one straight line can pass through the same two points.

22. *Def.*—The line mentioned in 21, corresponds to the axis or ground line in Chapters I. and II.; it will, however, in this

Chapter, be termed the *line of level;* and, unless the contrary be stated, its index will always be zero.

23. When two straight lines intersect in space, their plans will also cut each other, and the point of intersection will have the same index in both plans. It can therefore be ascertained whether two lines, whose plans have a common point, cut each other or not.

24. Since parallel straight lines are equally inclined to the plane of reference, their plans, which will be parallel, will have their indices increasing uniformly in the same direction. In other words, equidistant points on the plans of two parallel lines will have equidifferent indices. This affords a means of determining whether lines given by their plans are parallel or not.

25. *Def.—The contours* of a surface are the lines in which it is intersected by a series of equidistant horizontal planes.

From this definition it follows that the contours of a plane are a series of equidistant horizontal straight lines, parallel to the horizontal trace of the plane.

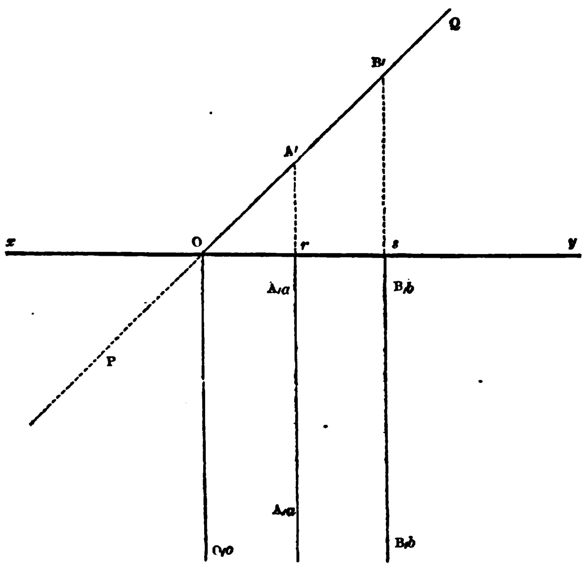
26. *Def.—The horizontals* of a plane are the plans of its contours, and are therefore a series of equidistant straight lines parallel to the horizontal trace of the plane.

27. A plane is adequately represented by two of its figured horizontals. This is evident; for if a line of level (22) be drawn at right angles to the horizontals it will be the horizontal trace of a vertical plane at right angles to the given plane, and to its contours. The traces of the contours on this plane may be at once determined, since they will be situated in the horizontals produced, and at distances from the line of level equal to their respective indices. These traces will be two points in the trace of the given plane; the straight line joining them will be that trace, and if produced to meet the line of level, the angle which it makes with that line will be the angle at which the plane is inclined to the horizontal plane. The straight line drawn through

the point in which the trace and line of level intersect, parallel to the horizontals, will be the horizontal trace of the plane.

To illustrate this, let $A_{,}a$ and $B_{,}b$ (Fig. 45) be two horizontals of a plane. Draw $x\,y$ at right angles to these horizontals, and meeting them in r and s respectively, make $r\,A'$ equal to a, $s\,B'$ equal to b; then the straight line P Q passing through A' and B', and meeting $x\,y$ in O, will be the trace of the given plane; the angle Q O y will be its inclination to the horizontal plane; and the straight line $O_{,}o$, drawn through O, parallel to $A_{,}a$ and $B_{,}b$, will be the horizontal trace of the plane.

Fig. 45.



This construction will sometimes be called "making an elevation of the given plane on the line of level $x\,y$," because the line P Q contains the elevations of all points and lines in the given plane, when the vertical plane, whose trace is $x\,y$, is considered as the plane of projection.

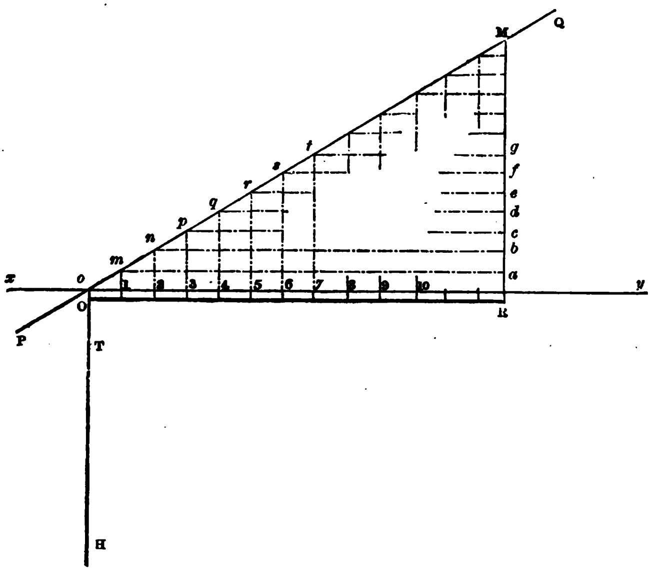
28. The horizontals of parallel planes will be parallel (*Euc.* XI.

16), and their indices will increase alike in the same direction. That is, equidistant horizontals in the two planes will have equidifferent indices. This property affords a means of ascertaining whether planes are parallel or not.

29. *Def.—The scale of a plane* is a straight line, perpendicular to its horizontal trace, graduated at the points in which it would be cut by equidistant horizontals of the plane, the points of graduation having attached to them the indices of the corresponding horizontals.

Let H T (Fig. 46) be the horizontal trace of a given plane. Draw $x y$ at right angles to H T, and meeting it in O. Through

Fig. 46.



O draw P Q, making with $x y$ an angle equal to the inclination of the given plane to the horizontal plane. From any point M in P Q draw M R perpendicular to $x y$; set off, on R M, R a = $a b = b c = c d$ = &c. &c. &c. = 1 unit of the scale of the draw-

ing; draw $a\,m$, $b\,n$, $c\,p$, $d\,q$, &c. &c. &c. parallel to $x\,y$, and meeting P Q in m, n, p, q, &c. respectively; draw through m, n, p, q, &c. straight lines parallel to M R, and meeting $x\,y$ in the points marked 1, 2, 3, 4, &c.; $x\,y$ will be the scale of the plane graduated according to the definition. This will be evident if the profile plane Q O R be supposed to take a vertical position.

30. The scales of parallel planes will be parallel, since they are perpendicular to the horizontals which are parallel (28).

31. If P Q be the elevation of a straight line whose plan is $x\,y$; the graduated scale O R will evidently be the scale of the line. In order to distinguish between the scale of a plane and that of a line, it is usual in practice to draw the scale of a plane as shown in the figure; whereas the scale of a line is represented by a single graduated straight line only. The scale should if possible be constructed in such a position as to have all points in the drawing on one side of it, and this side should be that opposite to the heavy line.

32. When the term "inclination" is used, without any qualification, it must be understood to signify the inclination of a line, or plane, to the plane of reference.

PROBLEM I.

Through a given point to draw a straight line having a given inclination.

Let A, a (Fig. 47) be the given point. Through A, in the horizontal plane, draw a straight line $x\,y$: on $x\,y$ as a line of level, make an elevation of A by drawing A A′ perpendicular to $x\,y$, and making A A′ equal to a. At the point A′ in A A′ make the angle A A′ O equal to the complement of the given inclination. Let A′ O cut $x\,y$ in O: the angle A O y being equal to the given angle, A′ O is the line required. O is a point in its plan having an index zero (17): the indices of two points in the line being thus known the line is determined (19) if the position of $x\,y$ is known. Otherwise the problem admits of an infinite

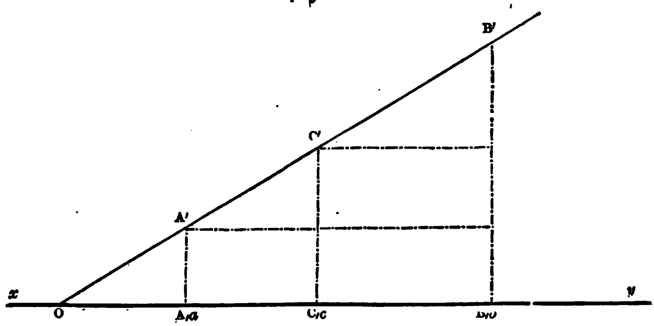
number of solutions; since all straight lines passing through A′, and meeting the horizontal plane at a distance from A equal to A O will have the same inclination. (Ch. I. 33, *Cor.* II.)

Problem II.

Given the plan of a line; to determine, (1) its inclination; (2) the length of the line between two points whose indices are given; (3) a point in the plan having a given index; (4) the index of a given point.

1. Let A,*a* and B,*b* (Fig. 47) be the two given points in the plan *x y*. Assuming *x y* as a line of level, draw A A′ equal to *a*, and B B′ equal to *b*, perpendicular to *x y*. Join B′ A′ and produce it to meet *x y* in O: the angle B O *y* will be the inclination required. (Chap. I. 33.)

Fig. 47.



2. The line, of which A B is the plan, is the hypothenuse of a right angled triangle, whose base is equal to A B; and whose perpendicular is $a \sim b$; A′ B′ is therefore the line required. If *a* and *b* had opposite signs, A A′ and B B′ would have been drawn on opposite sides of *x y*. The perpendicular would then have been $a + b$. This construction applies to (1) also.

3. Let C,*c* be the given point; draw C C′ perpendicular to *x y*;

through A′, B′, C′ draw straight lines parallel to $x\,y$: then by similar triangles,

$$a \sim b : a \sim c :: \text{A}'\,\text{B}' : \text{A}'\,\text{C}'.$$
$$:: \text{A B} : \text{A C} \quad (\alpha);$$

therefore, $\text{A C} = \frac{a \sim c}{a \sim b} \times \text{A B}$, which determines A C, since a, b, c and A B are known.

4. $\text{A B} : \text{A C} :: a \sim b : a \sim c \quad (\beta)$;

therefore, $a \sim c = (a \sim b) \times \frac{\text{A C}}{\text{A B}}$, which added to the index of A gives the index of C.

These constructions have been based on the assumption, that the index of the horizontal line A B is zero. Should this not be the case, the index of that line will have some definite value as $\pm d$, which must in every case be added, with its proper sign, to the indices obtained as above.

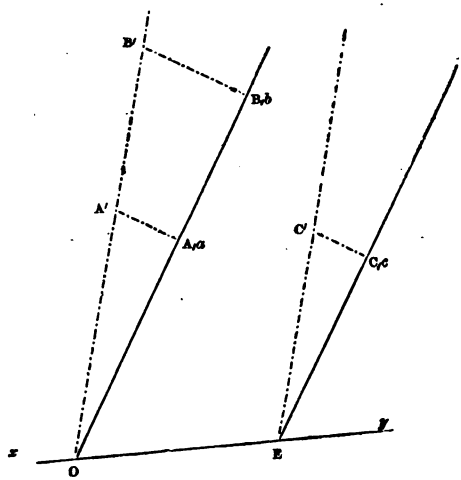
N. B. The readiest way of finding A C in (α) and $a \sim c$ in (β) will be to use the line of lines on the sector. (Part I. Chap. II.)

Problem III.

Through a given point to draw a straight line parallel to a given straight line.

Let A,a and B,b (Fig. 48) be two points in the plan of the given line: C,c the given point. Then since the lines are to be parallel their plans will be parallel (Chap. I. 11.) Draw through C a straight line P Q parallel to A B: P Q will be the plan of the required line. On A B as a line of level, make elevations A′ and B′ of A and B, as in Prob. I. Join A′ B′, and produce it to meet A B in O; O will be the zero point of the given line: make an elevation of C, as C′: through C′ draw C′ E, parallel to A′ B′, and meeting P Q in E: E will be the zero point of the required line. C′ E C its inclination, which is equal to that of the given line. The line is determined since the indices of two points in its plan are known.

Fig. 48.

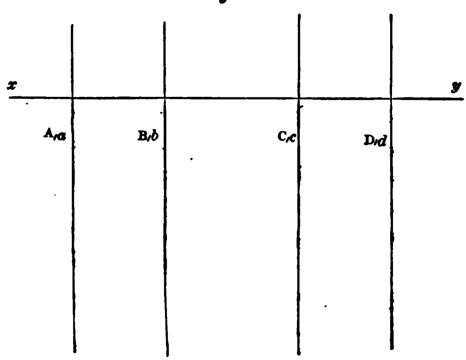


PROBLEM IV.

Through a given point to draw a plane parallel to a given plane.

Since the planes are to be parallel, their horizontals will be parallel. (*Euc.* XI. 16.)

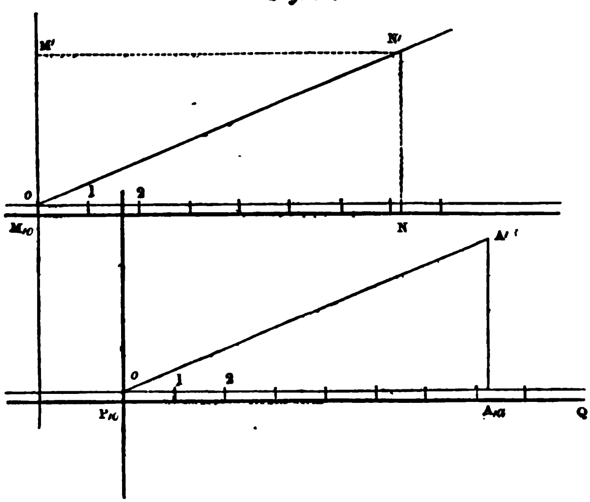
Fig. 49.



Let A,*a* and B,*b* (Fig. 49) be two horizontals of the given plane: C,*c* the given point. Through C draw *x y*, as a line of level, perpendicular to the horizontals. C,*c* will be a horizontal of the required plane. Make C D equal to A B: through D draw a straight line parallel to C,*c*: this will be another horizontal of the required plane; and its index *d* will evidently be equal to $c \pm (a \sim b)$, according as B is greater or less than A.

Otherwise, let M N (Fig. 50) be the scale of the given plane, the index of M being zero: A,*a* the given point. Through A

Fig. 50.



draw P Q the scale of the required plane, parallel to M N (28). Draw M M′ perpendicular to M N and equal to *a*: make the angle N M N′ equal to the inclination of the given plane: draw M′ N′ perpendicular to M M′ and meeting M N′ in N′: draw N′ N perpendicular to M N. Make A P equal to M N: then the index of P will be zero: and the plane is therefore determined, since its scale can be graduated from that of the given plane.

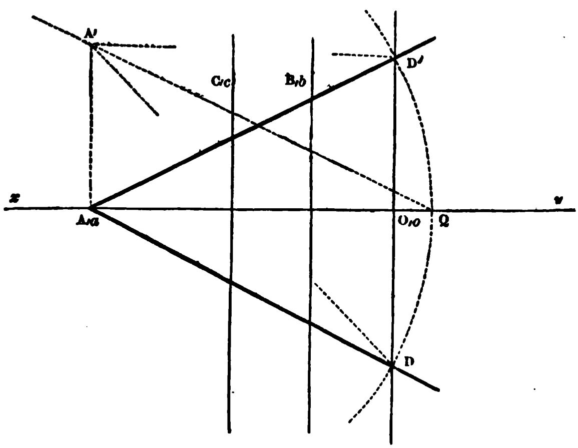
PROBLEM V.

Through a given point in a given plane, to draw a straight line that shall have a given inclination.

The inclination of the line cannot be greater than that of the plane. (Chap. I. 33.)

If the inclination of the line be equal to that of the plane; only one line can be drawn to fulfil the condition. In this case, the problem will be solved by drawing through the given point a straight line, perpendicular to the horizontals of the plane: this will be the plan of the required line: and the points in which it cuts the horizontals will have the same indices as the horizontals.

Fig. 51.



If the inclination be less than that of the plane, the problem will be indeterminate, since two lines can be drawn fulfilling the conditions of the problem; that is, lying in the given plane, and making a given angle with the horizontal plane.

Through A,a, the given point (Fig. 51), draw $x\,y$ perpendicular to C,c and B,b, the horizontals of the given plane. On $x\,y$, as a

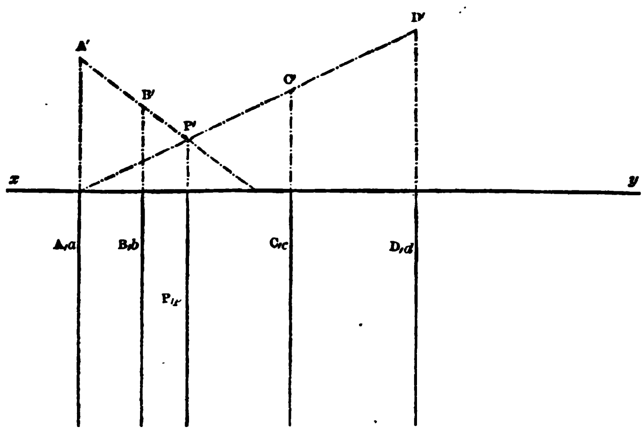
line of level, make an elevation of the given point $A_{,}a$, as A′ : find the point in $x\ y$, whose index is zero (Prob. II.), let it be O : through O draw O D, parallel to the given horizontals, this will be the trace of the given plane. At the point A′ in A A′ make an angle A A′ Q, equal to the complement of the proposed inclination. Let A′ Q cut $x\ y$ in Q ; A′ Q will be a line having the proposed inclination : A Q its plan. With A as a centre and radius A Q, describe a circle cutting D D′ in D and D′: join A D and A D′ ; these lines will be the plans (Chap. I. 33) of two lines fulfilling the conditions.

Problem VI.

To find the intersection of two given planes.

1. Let the horizontals $A_{,}a$, $B_{,}b$ and $C_{,}c$, $D_{,}d$ (Fig. 52) of the given planes be parallel. Draw $x\ y$ perpendicular to the hori-

Fig. 52.

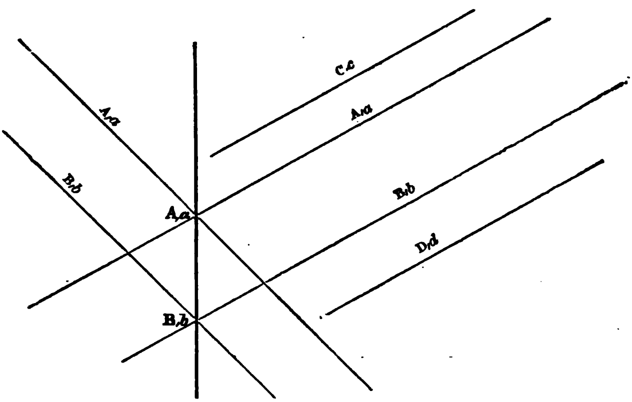


zontals ; on $x\ y$, as a line of level, make elevations A′ B′, C′ D′ of the planes. Through P′ the point in which these elevations intersect, draw $P_{,}p$, parallel to $A_{,}a$ or $C_{,}c$; $P_{,}p$ will be the plan

of the intersection of the planes, and its index p may be readily determined from the indices of the given horizontals. (Prob. II.)

2. If the horizontals of the two planes be not parallel, let A,*a* and B,*b* (Fig. 53) be the horizontals of one plane; C,*c* and D,*d* those of the other: in this second plane find (Prob. II.) two horizontals A,*a* and B,*b*: the points in which these respectively in-

Fig. 53.

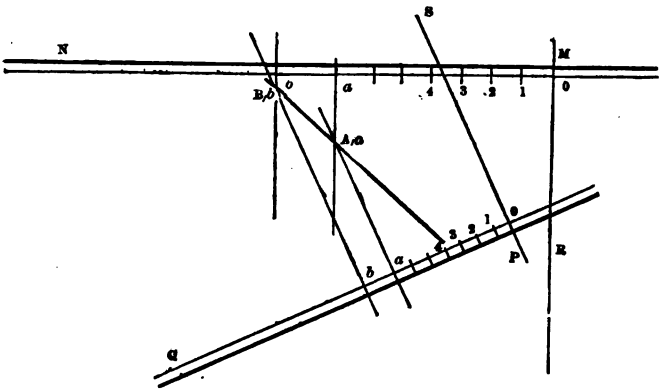


tersect the corresponding horizontals of the first plane, will be two points in the plan of the required intersection. The straight line drawn through these points will be the plan required: and, since the indices of two points in it are known, it is fully determined.

Otherwise, let M N and P Q (Fig. 54), be the scales of the given planes; M R and P S being their traces: through the points of the scales figured *a* and *b* in each draw straight lines parallel to the traces of the respective planes. Let these parallels intersect in A,*a* and B,*b*: the indefinite straight line passing through A and B will evidently be the plan of the intersection.

The solutions of the cases when the scales are parallel, and when one of the planes is perpendicular to the plane of reference, are left as an exercise.

Fig. 54.

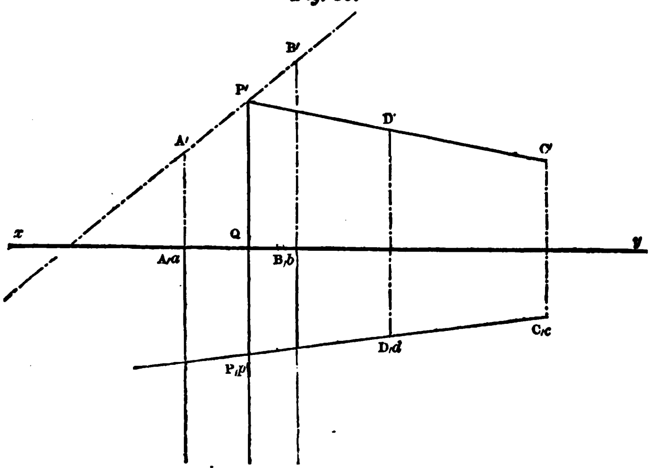


PROBLEM VII.

To find the intersection of a given straight line and a given plane.

Let A,*a* and B,*b* (Fig. 55) be the horizontals of the given

Fig. 55.



plane: D C the plan of the line. Draw $x\ y$ perpendicular to A,a or B,b: on $x\ y$ as a line of level, make elevations of the plane and the line.

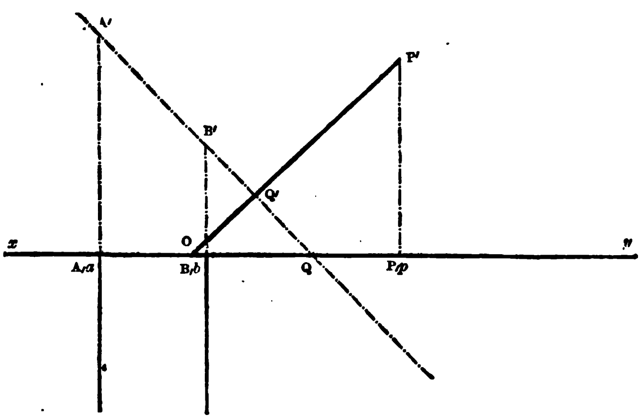
The point P′ in which C′ D′ the elevation of the line cuts A′ B′, the elevation of the plane will be the point required. For the elevation of the point in which the line meets the plane must be in A′ B′ and in C′ D′; it will, therefore, be P′, the point in which these lines intersect. Let P′ P drawn parallel to D D′, cut C D in P: P will be the plan of the point of intersection: its index p being equal to P′ Q.

Problem VIII.

To draw a straight line through a given point perpendicular to a given plane.

Let A,a, B,b (Fig. 56) be two horizontals of the given plane; P,p the point. Since the line is to be perpendicular to the plane, its plan will be perpendicular to the trace of the plane (*Euc.* XI. 18, and Def. 3).

Fig. 56.



Let $x\ y$ be this plan, on it make an elevation A′ B′ of the plane; and an elevation P′ of the point. Then the elevation of

the line will be perpendicular to that of the plane (Chap. I. 30). Draw P′ Q′ perpendicular to A′ B′; this will be the elevation of the line; and since P′ Q′ lies in the vertical plane passing through $x\,y$, it is evident that P′ Q′ measures the distance of the given point from the given plane. If P′ Q′ meet $x\,y$ in O: P′O P will be the inclination of the line.

PROBLEM IX.

Through a given point to draw a plane perpendicular to a given line.

Let $x\,y$ (Fig. 56) be the plan of the line, $P_{,}p$ a point in it, $B_{,}b$ the given point. P′ O being the elevation of the line, B′ that of the given point; through B′ draw A′ B′ perpendicular to P′ O; A′ B′ will be the elevation of the plane: and since the indices of the horizontals passing through B and Q are known, the plane is determined, and may be easily represented either by its horizontals or by its scale.

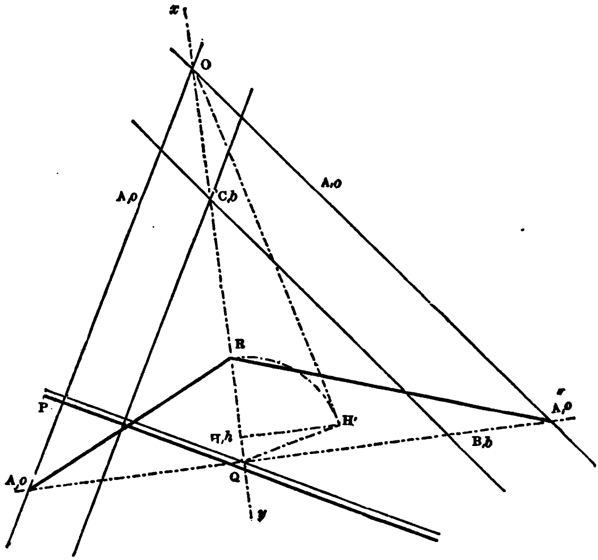
PROBLEM X.

To determine the angle between two given planes.

Determine the traces $A_{,}o$ of each plane and two horizontals $B_{,}b$: let these horizontals intersect in C, and the traces in O, (Fig. 57); the straight line $x\,y$ drawn through O and C will be the plan of the intersection of the planes (Prob. VI.). In $x\,y$ find a point $H_{,}h$; draw H H′ perpendicular to $x\,y$, and h units in length. Join O H′; from H′, draw H′ Q perpendicular to O H′: through Q draw A Q A″ perpendicular to $x\,y$: meeting the traces in A and A″. Make Q R equal to Q H′: join A R and A″ R: the angle A R A″ will be the angle required. (Chap. I. 35.)

This problem may be solved in a similar way when the planes are given by their scales.

Fig. 57.



PROBLEM XI.

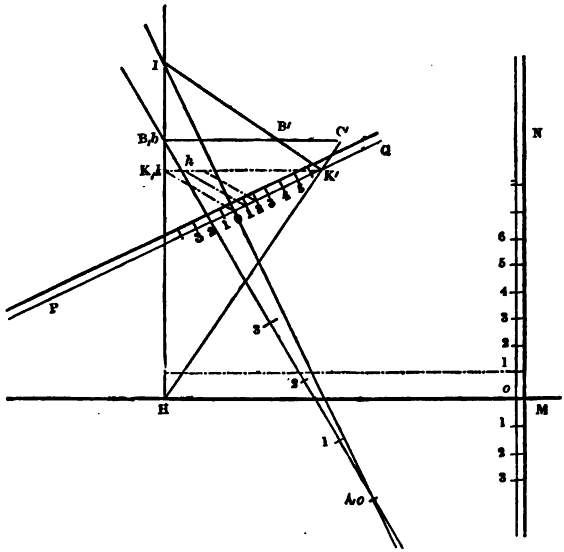
Through a given straight line to draw a plane perpendicular to a given plane.

If from any point in the given line a straight line be drawn perpendicular to the given plane. The plane containing this perpendicular and the given line will be the plane required.

Let M N (Fig. 58) be the scale of the given plane; H M being its trace, A B the plan of the given line; the index of A being zero, that of B, *b*. Through B draw H I perpendicular to H M: B C′ parallel to H M: make B B′ equal to *b*; and B C′ equal to the index of the corresponding point of the plane. Join C′ H, and draw K′ I through B′ perpendicular to C′ H, and meeting H I in I; I will be the trace of this perpendicular (Prob. VIII). The trace of the required plane must pass through I and also through A, the trace of the given line. I A is there-

fore the trace required. Draw K′ K parallel to H M: K K′ will be equal to the index of a point in the plane. If the scale line P Q be drawn perpendicular to the trace I A, and graduated, the problem is solved.

Fig. 58.



To graduate the scale: let I A and P Q intersect in 0: join K 0: in K K′ make K *h* equal to one unit of altitude: draw *h* 1 parallel to K 0 and meeting P Q in 1; 0 1 will be the first division of the scale; which may therefore be completed by setting off along P Q distances equal to 0 1, and annexing the proper indices.

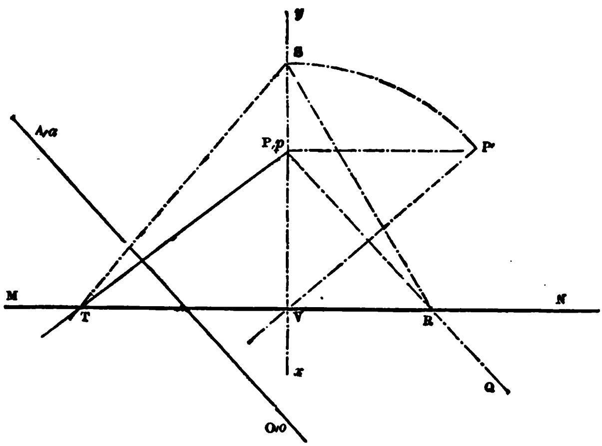
Problem XII.

Through a given point in a given plane, to draw a straight line, which shall make a given angle with a given straight line in that plane.

Let M N (Fig. 59) be the trace of the plane, A O the figured

plan of the given line; $P_{,}p$ the given point: through P draw P Q, parallel to A O, meeting M N in R, and $x\ y$ perpendicular to M N; draw P P′ perpendicular to $x\ y$, and equal to p units of altitude. Join P′ with V the point in which $x\ y$ cuts M N;

Fig. 59.



make V S equal to V P′ : join R S; make the angle R S T equal to the given angle. Join T P; T P will be the plan of the required line, which may be figured by drawing parallels to M N.

This construction is obvious, since the triangle T P R is evidently the plan of a triangle identical with T S R. (Chap. I. 35.)

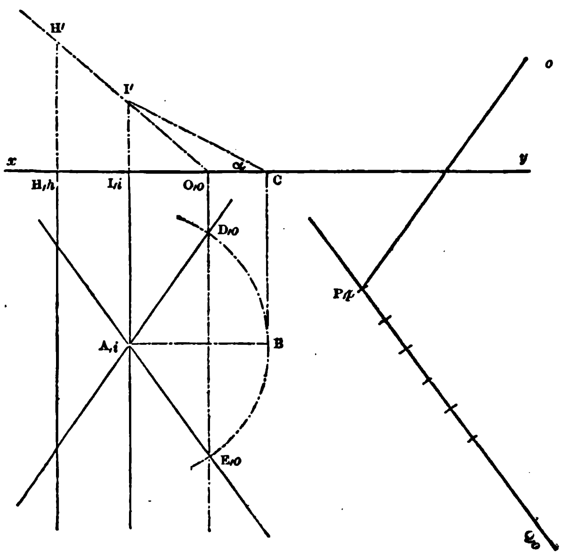
PROBLEM XIII.

Through a given point to draw a straight line, having a given inclination and parallel to a given plane.

Let $H_{,}h$, $I_{,}i$ (Fig. 60), be two horizontals of the given plane, $P_{,}p$, the given point; on $x\ y$ as a line of level at right angles to the horizontals, make an elevation H′ I′ of the plane, and let

H′ I′ meet $x\ y$ in O: through O draw O E perpendicular to $x\ y$; O E will be the trace of the plane. At I′ make the angle I I′ C, equal to the complement of the given inclination. From any point A in the horizontal $I_{,}i$, draw A B perpendicular to $I_{,}i$, and

Fig. 60.



equal to I C: with A as a centre and radius A B, describe a circle cutting the trace O E in D and E: D and E will be the zero points of two lines in the plane having the given inclination: the plans of these lines will evidently be A D and A E. Through $P_{,}p$ draw P Q and P R parallel to A E and A D respectively; P Q and P R will be the plans of two lines fulfilling the conditions of the problem. The index of P and the inclination of the lines being given, the plans may readily be figured.

N.B. The inclination of the line must not be greater than that of the plane (Chap. I. 33). The problem, depending upon the intersection of a straight line and a circle, will evidently admit either of two solutions, of one solution, or be impossible, accordingly as the circle cuts O D,

touches O D, or does not meet O D: the third case arises when the inclination of the line is greater than that of the plane; the second when it is equal to it; the first, when it is less.

Problem XIV.

To draw a plane, making a given angle with a given plane, and containing a given line in that plane.

The line will evidently be the intersection of the given and required planes.

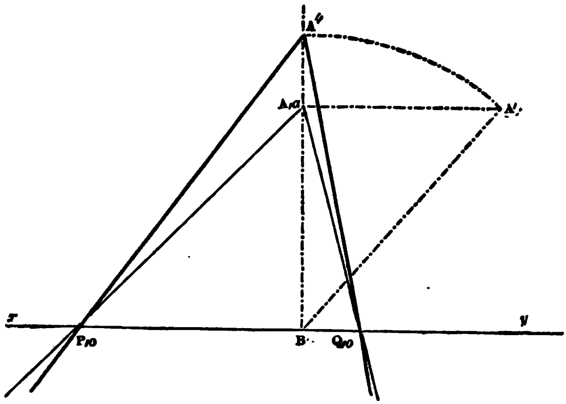
Let A,o (Fig. 57) be the trace of the given plane, P Q its scale, $x y$ the plan of the given line; in $x y$, take a point H,h, draw H H′ perpendicular to $x y$ and equal to h units of altitude; join H′ with O the trace of the line; draw H′Q perpendicular to O H′ and meeting $x y$ in Q; through Q draw A Q A″ perpendicular to $x y$; make Q R equal to Q H′; join A R; draw R A″, making with A R an angle equal to the profile angle of the two planes, and cutting AA″ in A″; A″ will be a point in the trace of the required plane, and O is another point in the trace, consequently, the straight line A″ O is the trace; and since H,h is a point in the plane, if the scale be drawn through H perpendicular to the trace, it can be graduated at once.

Problem XV.

Given the plans of two lines to determine the angle contained by the lines.

Let A P and A Q (Fig. 61) be the plans of the lines, P and Q being their traces, A,a the point of intersection; then the straight line P Q will be the trace of the plane containing the lines. Through A draw A B perpendicular to P Q; draw A A′ perpendicular to B A, and equal to a units of altitude; in B A produced, make B A″ equal to B A′, join P A″ and Q A″; the angle P A″ Q will be the angle required (Chap. I. 35).

Fig. 61.

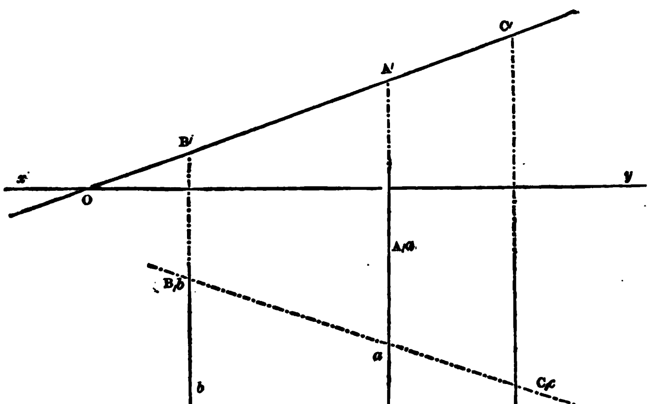


PROBLEM XVI.

To determine the horizontals and the inclination of a plane containing three given points.

Let A,a, B,b, C,c (Fig. 62) be the given points.

Fig. 62.



Join any two of these points as B and C; in the line B C, find a point whose index is a (Prob. II.). Join this point with A, a: the line thus drawn will be a horizontal of the required plane, having an index a; a line drawn through B or C parallel to A, a will be a second horizontal of known index, and the plane is therefore determined.

Draw $x\ y$ at right angles to these horizontals; on it as a line of level make elevations A′, B′, C′ of A, B, C respectively. A′, B′, C′, being points in the vertical trace of a plane, should be in a straight line. The angle C′ O y which this line makes with $x\ y$ is evidently the inclination of the plane.

Problem XVII.

To determine the plan of a rectilineal figure, having given (1) the inclination of its plane, and that of one side: (2) the inclination of two of its sides.

Either of these data will suffice to determine the position of the figure with reference to the horizontal plane. The problem will be simplified by assuming the plane of the figure at right angles to the vertical plane of projection; and, consequently, its trace perpendicular to the ground line, or line of level.

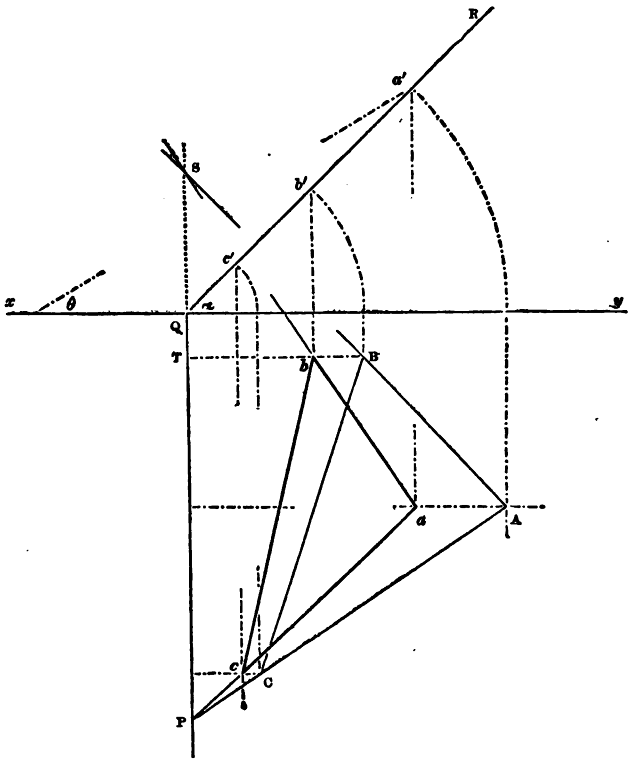
1. Let P Q R (Fig. 63) be the given plane, draw a P, the plan of a line in it, having the given inclination (see Prob. XIII.). Find, by Chap. I. 35, the position A, of the point whose plan is a, when the plane P Q R has been turned about its trace P Q into coincidence with the horizontal plane. Make A B equal to the side whose inclination is given; on A B describe the figure A B C in its real magnitude; find its plan $a\ b\ c$, by Chap. II., Prob. XXXVI, as shown in Fig. 63.

Note.—It is manifest that any line A B, and its plan $a\ b$, if produced, will meet the trace P Q in the same point S. The point b may, therefore, be found by drawing B T parallel to $x\ y$, and cutting a S in b; this method often considerably shortens the construction. Although introduced here as a separate problem,

this problem is merely a combination of Prob. XIII., Chap. III., with Prob. XXXVI., Chap. II.

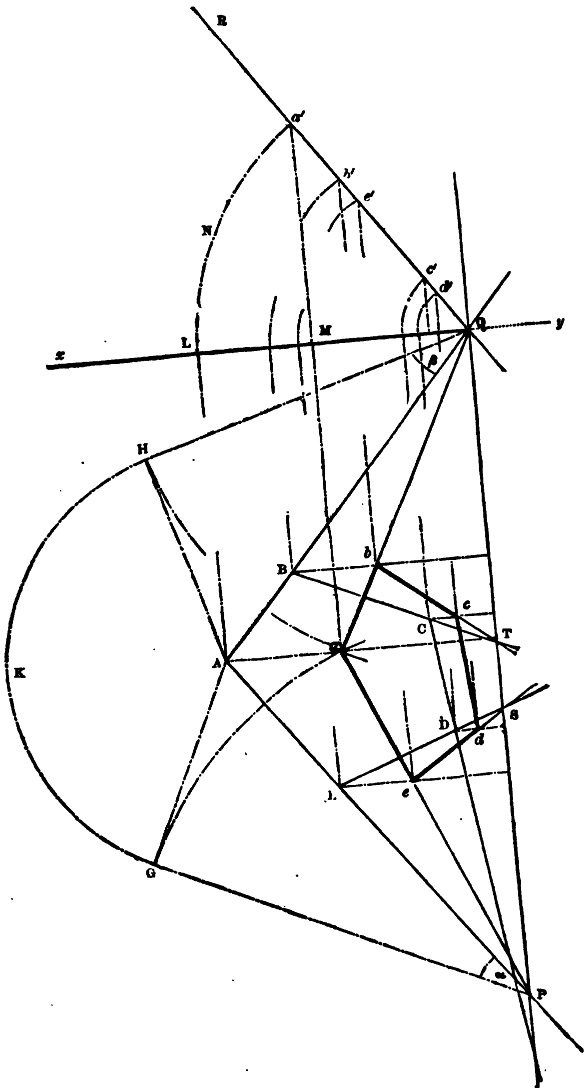
2. Let A B C D E (Fig. 64) be the given figure, its two sides A B and A E being inclined α° and β° respectively to the

Fig. 63.



horizon. In A E produced assume any point P, as the trace of A E; make the angle A P G equal to α; from A draw A G perpendicular to P G: with A as a centre and radius A G, describe the circle G K H; draw Q H touching this circle in H, making an angle β with A B, and meeting A B, or A B produced, in Q; Q will be the trace of A B; and the straight line P Q the trace

Fig. 64.



of the plane containing the given figure. Then, by Chap. I. 35, the plan of A will be in the straight line A T, drawn from A perpendicular to P Q; its distance from P Q being equal to the base of a right-angled triangle, whose hypothenuse is A T and perpendicular A H. This triangle may be constructed in the following manner:—With P, or Q, as a centre, and radius P G, or Q H, describe a circle cutting A T in *a*, *a* will be the plan of A. Through Q draw $x\ y$ perpendicular to P Q; $x\ y$ may be assumed as the trace of a vertical plane perpendicular to the plane of the figure. Draw A L perpendicular to $x\ y$; with centre Q, and radius Q L, describe a circle L N a'; draw a M perpendicular to $x\ y$, and meeting this circle in a'; a' will be the elevation of A on the vertical plane, whose trace is $x\ y$. Through a' draw Q R; Q R will be the vertical trace of the plane of the figure; the angle R Q x its inclination to the horizon, and Q M a' the triangle required. The elevations, b', c', d', e', and the plans b, c, d, e may now be readily found (Chap. I. 35); or by the principle stated in the preceding Note. Both construetions are shown in Fig. 64.

Obs.—The remaining portion of this Chapter will require a knowledge of the projections of a cone and its tangent plane; which may be acquired from Chap, IV., Probs. XI. XII. and XIII.

Problem XVIII.

To draw a plane to contain a given line and have a given inclination.

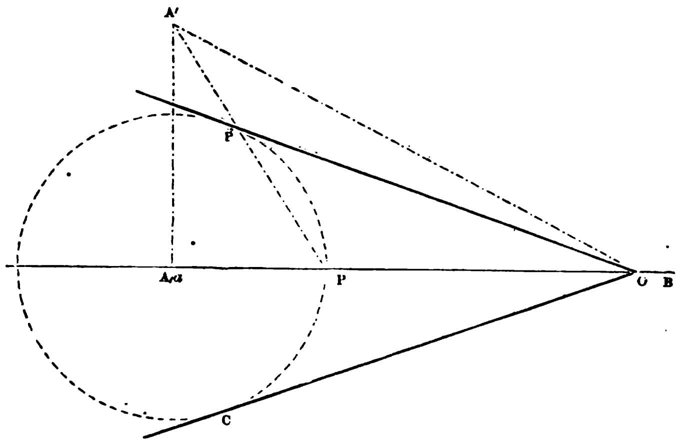
1. If the line be horizontal, it will be one of the horizontals of the required plane.

2. If the line be not horizontal, let A B (Fig. 65) be its plan; on A B, as a line of level, make an elevation A′ of A.

At A′ in A A′, make the angle A A′ P equal to the complement of the given angle of inclination. Let A′ P cut A B in P; with A as a centre, and radius A P, describe a circle C P D; from O, the zero point of A B, draw O C and O D, touching this circle in C and D. O C and O D will be the horizontals of two planes

fulfilling the proposed conditions. The inclination of the plane being given other horizontals may be determined, having given indices.

Fig. 65.



NOTE.—The inclination of the plane must not be less than that of the line (Chap. I. 33.) If the inclination of the plane be equal to that of the line, there will be only one solution, P and O will coincide, and the horizontal trace of the plane will pass through O and be perpendicular to A B.

PROBLEM XIX.

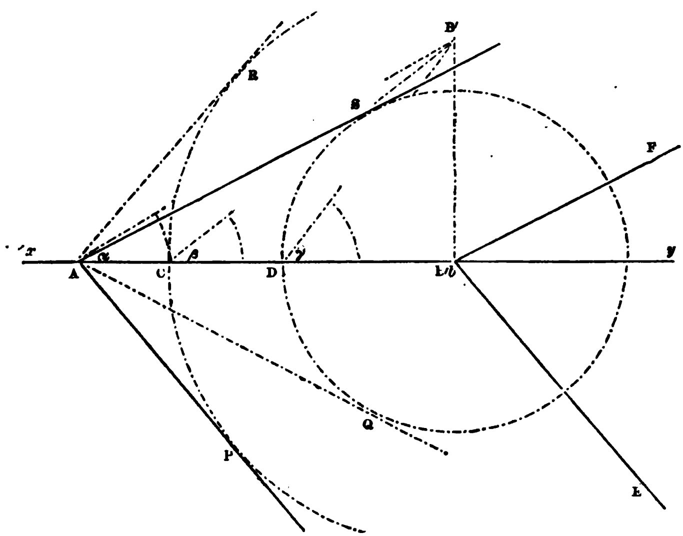
Through a straight line inclined at a given angle, to draw two planes inclined at given angles.

Let the angle of inclination of the line be a°; those of the planes being β° and γ° respectively.

Assume the line to be projected in $x\,y$ (Fig. 66) at any point A in $x\,y$; make the angle B A B′ equal to a, and from a second point B draw B B′ perpendicular to $x\,y$; A will be the trace of a line having the proposed inclination; B,b (if B B′ be equal to b) will be a point in both planes.

Make the angle B C B′ equal to β, and B D B′ equal to γ: B′b may be assumed as the common vertex of two right cones whose generatrices are inclined β° and γ° respectively to the horizon: the required planes must be tangents to these cones, each to each; their traces must pass through A, and touch the traces of the cones (14 and 15, Chap. III.).

Fig. 66.



The following will therefore be the construction :—

With B as a centre and radii B C, B D describe the circles P C R, Q D S: from A draw A P and A S touching these circles in P and S: A P and A S will be the traces of two planes fulfilling the conditions. Through B draw B E parallel to A P; B F parallel to A S; these lines will be horizontals of the planes, having an index b the planes are therefore determined.

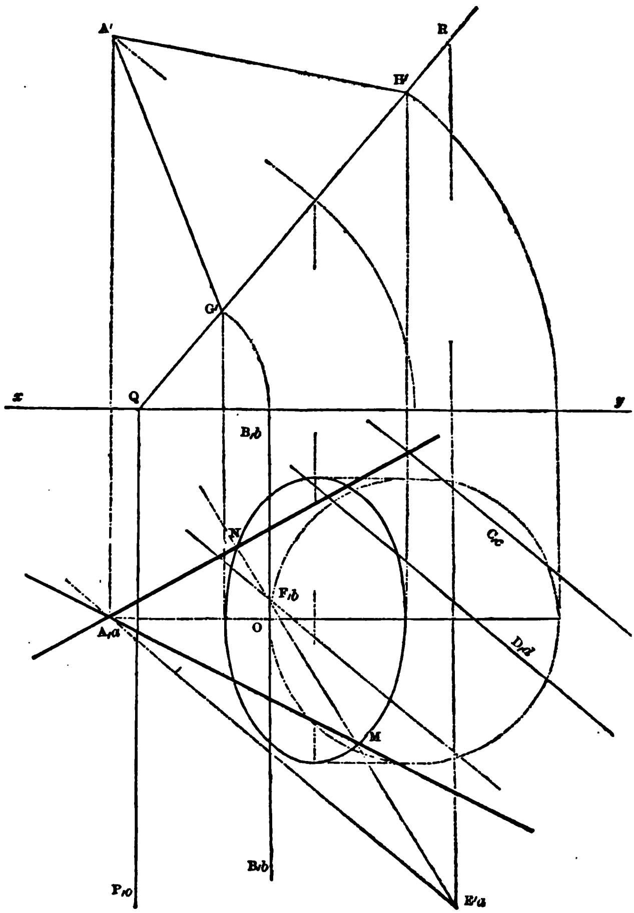
This problem admits of four solutions, as shown in the figure.

Cor. If the angles α, β (or γ) and B A S, were given, the angle γ (or β) might readily be found, as is evident from the construction.

PROBLEM XX.

To draw a straight line through a given point to make a given angle with one given plane and to be parallel to another given plane.

Fig. 67.



The angle between the line and the plane must not be greater than that contained by the two planes. (Chap. I. 33.)

Let A,*a* (Fig. 67) be the given point; P Q the trace, B,*b* a horizontal of the first plane; C,*c* and D,*d* two horizontals of the second plane. Draw $x\ y$ perpendicular to P Q, and on $x\ y$ as a line of level make an elevation Q R of the first plane: assume A,*a* to be the vertex of a right cone, with circular base, standing on the plane P Q R; so that A′ G′ H′ is its elevation; the curve M O N being the plan of the base of the cone (Chap. II. Prob. XXXVII); and the angle A′ G′ H′ equal to the given angle.

Through A,*a* draw a plane parallel to the second (Prob. IV.): find (Prob. VI.) E F the intersection of this plane with the plane P Q R: let E F meet the plan of the base of the cone in M and N: through A draw A M and A N: these will be the plans of two lines fulfilling the conditions of the problem. For being the plans of the generatrix of the cone in two positions, they make the given angle with the plane P Q R, and also being in a plane parallel to the second plane, they are parallel to that plane.

PROBLEM XXI.

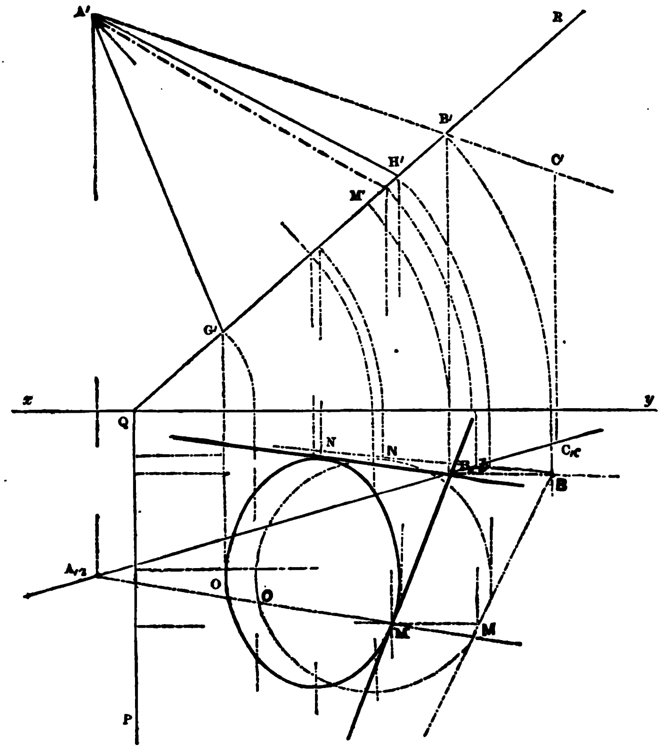
To draw a plane to contain a given line, and to make a given angle with a given plane, not containing the line.

Let P Q (Fig. 68) be the trace of the plane, A C the plan of the line. Draw $x\ y$ at right angles to P Q; and on $x\ y$ as a line of level make an elevation Q R of the given plane.

Make any point A,*a* in the given line, the vertex of a right cone with circular base standing on the plane P Q R; its generatrix making with that plane an angle equal to the given angle. Let A′ G′ H′ be the elevation of that cone, the ellipse M O N being the plan of its base. Find by Problem VII. B,*b* the point in which the line (A C, A′ C′) meets the plane P Q R. From B draw B M and B N, touching M O N in M and N respectively: B M and B N will be the plans of two lines in which the plane P Q R is cut by two planes, which will fulfil the conditions of the problem, when made to contain the point A *a*:

which may readily be accomplished by determining the trace of the plane by means of those of the lines (A B, A′ B′) and (A M, A′ M′).

Fig. 68.



By turning the plane P Q R, with the base of the cone in it, above its trace P Q, as shown in the figure, and drawing tangents B M, B N to the circle M O N, the points M and N may be found (Chap. I. 35), without the necessity of describing the ellipse M O N and drawing tangents to it.

PROBLEM XXII.

To draw a plane that shall have a given inclination, and make a given angle with a given plane.

The solution of this problem will merely be indicated without the aid of a diagram, which the preceding portion of the work should enable the reader to supply.

Make any point the common vertex of two right cones A and B, with circular bases; A having its axis perpendicular to the horizontal plane and its generatrix making with that plane an angle equal to the given inclination; B having its axis perpendicular to the given plane, its generatrix making with that plane an angle equal to the angle between it and the required plane.

Determine the intersections of these cones by the horizontal plane; that of A will be a circle; that of B an ellipse. Every common tangent that can be drawn to the circle, and the ellipse will be the trace of a plane fulfilling the given conditions: and since the inclination of such plane is given, its scale can be at once constructed.

It is evident that, when the circle and the ellipse fall entirely without each other, four common tangents can be drawn; when they touch, three; when they cut, two; when one falls within the other, none. The problem, therefore, admits of four solutions; three solutions; two solutions; or it is impossible.

EXERCISES.

1. Draw a plane inclined 56° to the horizon; in this plane place a straight line inclined 46° to the horizon. From any point in this line erect a perpendicular to the plane 2 inches long.

2. Draw two planes inclined at angles of 40° and 63° respectively, and intersecting in a line inclined at 27°. Find the dihedral angle contained by the planes. Scale, 10 units to the inch.

3. Two lines, each two inches long, are inclined at 25° and 30° respectively to the horizon in a plane inclined at an angle of 50°. Draw the plan of the lines in position, and construct the real angle contained by them. Scale, 10 units to an inch.

4. Determine the inclination to the horizon of a line 2·78 inches long, one extremity being 1·56 inches higher than the other. Also of another line 3·6 inches long, its plan being 2·3 inches long.

5. A line, 2 inches long, has its sides respectively 5 and 19 units above the horizontal plane. Find its intersection with a plane inclined at 60° to the horizon, assuming any relative position on plan. Scale, 10 units to one inch.

6. Find the intersection of two planes inclined at 30° and 54° to the horizontal; their horizontals being parallel.

7. The plans of two lines contain an angle of 80°, the lines being inclined to the horizon at angles of 50° and 35°. Determine the real angle contained by the lines, and also the inclination of the plane in which they lie. Through the line inclined at 35°, draw a plane making an angle of 75° with the plane containing the two lines.

8. Draw a plane, making an angle of 42° with another inclined at 50° to the horizon; and passing through a line, not in the given plane, inclined at 30° to the horizon.

9. Draw a line inclined 35° to the horizon; through it draw a plane inclined at 58°; and in the latter place a line making an angle of 30° with the former line.

10. Draw a plane inclined 42° to the horizon, and a perpendicular to it 2 inches long: through the perpendicular draw a plane, making an angle of 55° with the horizon. Find by construction the dihedral angle of the planes.

11. A plane inclined at 60° to the horizon makes with another plane an angle of 70°, the intersection of the two planes being inclined at 42°. Find the inclination of the second plane.

12. An equilateral triangle, with a side of 2·2 inches, has two of its sides inclined at 25° and 40° to the horizon. Draw its plan. Determine the inclination of the plane of the triangle. Draw the plan of the circle circumscribing the triangle.

13. Through a line inclined at 40° to the horizon, draw a plane, making an angle of 30° with a plane inclined at 25° to the horizon. Scale, 10 units to an inch.

14. An equilateral triangle of 3 inches side rests on one angle and has the sides containing that angle inclined at 20° and 30° respectively to the horizon. Construct its plan.

15. Construct the plan of a square of 2 inches side resting on a plane inclined at 50° to the horizon, one of the sides being inclined at an angle of 35° to the horizon.

16. Draw the plan of an isosceles triangle having a base of 1·5 inches and sides of 2·25 inches; the triangle being so placed that the base is inclined 23°, and the line joining one end of the base with the middle point of the opposite side 51°, to the horizon.

CHAP. IV.

ELEMENTARY PROBLEMS ON THE PROJECTION OF SOLIDS.

1. *Def.*—A *polyhedron* is a solid figure bounded by plane rectilineal figures. There are only five *regular* polyhedrons (that is, polyhedrons which have all their faces equal regular polygons, and all their solid angles equal); these are the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron. The projections of four of these will be given here.

For definitions of these and other solids mentioned in this Chapter, see *Euc.* XI. Definitions.

2. The determination of the projection of a solid, when a polyhedron, will evidently consist in finding the projections of its edges by means of those of its angular points. Should, however, any of the superficies bounding the solid be contained by curved lines, the projections of such lines must be constructed in a manner similar to that shown in Chap. II. Problem XXXVII.

PROBLEM I.

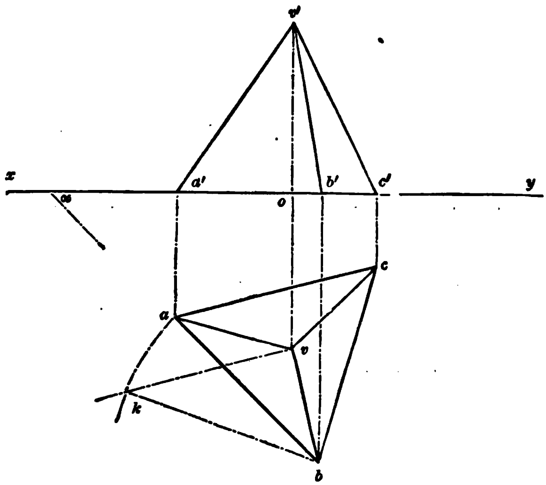
To construct the projections of the tetrahedron.

Assume the solid to be resting on one of its faces on the horizontal plane.

Let $a\,b$ (Fig. 69) be one of the edges resting on the horizontal plane, and inclined at any given angle α to $x\,y$; on $a\,b$ describe the equilateral triangle $a\,b\,c$; find v the centre of the circumscribing circle; join $a\,v$, $b\,v$, $c\,v$; v will be the plan of the vertex;

a b c will be that of the base; *a v*, *b v*, and *c v*, those of the edges meeting in *v*.

Fig. 69.



From *v* draw *v k* perpendicular to *v b*; with *b* as a centre, and radius *b a*, describe a circle cutting *v k* in *k*; *k v* will evidently be the perpendicular height of the pyramid. Draw *a a'*, *b b'*, *c c'*, *v v'* perpendicular to *x y*; make *o v'* equal to *v k*; join *a' v'*, *b' v'*, *c' v'*, the figure thus formed will be the elevation of the solid.

PROBLEM II.

To construct the projections of the octahedron.

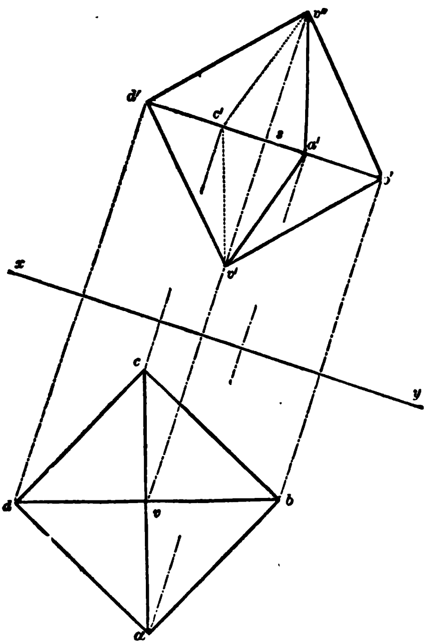
Assume the axis of the solid to be vertical.

Let *a b* (Fig. 70) be one edge of the base; on *a b* describe the square *a b c d*; draw the diagonals *a c*, *b d*, cutting each other in *v*; the figure thus drawn will be the plan of the solid.

From *v* draw *v v''* perpendicular to *x y*; make *v' v''* equal to

the diagonal of the square $a\ b\ c\ d$; bisect $v'\ v''$ in s; draw $d'\ b'$ parallel to $x\ y$, and $a\ a'$, $b\ b'$, $c\ c'$, $d\ d'$ perpendicular to $x\ y$; v', v'', a', b', c', d', will be the elevations of the angular points of the

Fig. 70.



solid, whose elevation will be formed by joining $v''\ d'$, $v''\ a'$, $v''\ b'$, $v'\ d'$, $v'\ a'$, $v'\ b'$: the elevations $v''\ c'$ and $v'\ c'$ would be invisible.

Problem III.

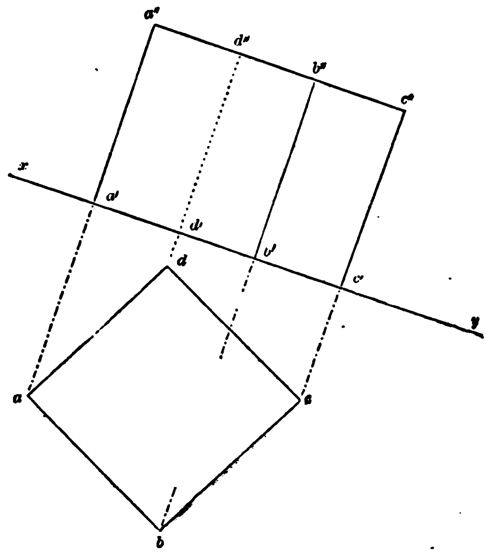
To construct the projections of the hexahedron, or cube.

Assume the solid to be resting on one of its faces on the horizontal plane.

Let $a\ b$ (Fig. 71) be one edge of that face. Describe on $a\ b$,

in the horizontal plane, the square $a\,b\,c\,d$; this will be the required plan.

Fig. 71.



Draw $a\,a'$, $b\,b'$, $c\,c'$, $d\,d'$, perpendicular to $x\,y$, in these lines produced make $a'\,a''$, $b'\,b''$, $c'\,c''$, $d'\,d''$, each equal to $a\,b$. The figure $a''\,c''\,c'\,a'$ will be the elevation of the solid. It would, perhaps, be simpler to make $a'\,a''$ equal to $x\,y$, and through a'' to draw $a''\,c''$ parallel to $x\,y$.

PROBLEM IV.

To construct the projections of the icosahedron.

Assume one axis of the solid to be vertical, and one of its edges parallel to the vertical plane of projection.

Let $a\,b$ (Pl. I. Fig. 6) be one edge of the solid; on $a\,b$ describe the regular pentagon ($a\,b\,c\,d\,e$, $e'\,a'\,d'\,b'\,c'$) in a horizontal plane,

so that $e' c'$ is parallel to $x y$. Determine a point (s, s') such that, joining it with the angular points of the pentagon $a b c d e$, five equilateral triangles will be formed. To effect this, s being the centre of the pentagon, describe on $s b$ the right angled triangle $s b n$, in which the hypothenuse $b n$ is equal to $b c$: $s n$ will be the perpendicular height of a pyramid whose base is the pentagon $(a b c d e, e' a' d' b' c')$. In a second horizontal plane describe the pentagon $g h k l m$ equal to and concentric with the former, but having its angular points respectively in the middle points of the arcs of the circumscribing circle subtended by the sides of the pentagon $a b c d e$. To determine the distance of this second horizontal plane from the former one, join $m e$, describe on it the right angled triangle $m e o$, having its hypothenuse $m o$ equal to $m l$: $e o$ will be equal to the vertical distance between the two planes.

Join $s c$, $s k$, $s d$, $s l$, $s e$, $s m$, $s a$, $s g$, $s h$, $a g$, $g b$, $b h$, $h c$, $c k$, $k d$, $d l$, $l e$, $m a$; this will complete the plan of the solid.

To finish the elevation, make $d' s'$ equal to $s n$; join $s' e'$, $s' a'$, $s' d'$, $s' b'$, $s' c'$; this will be the elevation of the pyramid whose base is $a b c d e$: in the perpendicular $e e'$ make $e' m'$ equal to $e o$, draw through m' a straight line parallel to $x y$ and meeting the perpendiculars from l, d, k, and h in l', g', k', h': $m' l' g' k' h'$ will be the elevation of the pentagon $g h k l m$: join $a' m'$, $a' g'$, $b' g'$, $b' h'$, $l' e'$, $l' d'$, $k' d'$, $k' c'$: this will complete the elevation of a zone bounded by ten equal equilateral triangles (constr.). The elevation of the solid may now be completed by drawing that of the pyramid whose base is $g h k l m$, and vertex $(s_{,}, s'_{,})$.

PROBLEM V.

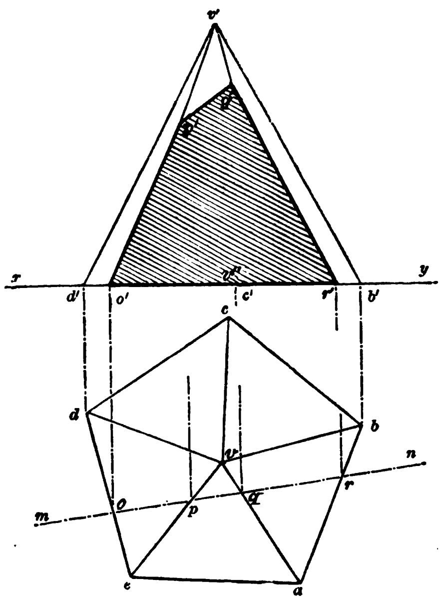
To construct the sectional elevation and plan of a pyramid standing on its base on the horizontal plane.

Let the pyramid be a pentagonal one, $b c$, one edge of the base being inclined at an angle a to $x y$ (Fig. 72); on $b c$ describe the regular pentagon $b c d e a$: find v the centre of the circumscribed circle: join $v a$, $v b$, $v c$, $v d$, $v e$: this will be the plan of the pyramid. Draw $v v'$ perpendicular to $x y$ and make $v'' v'$ equal to the perpendicular height of the pyramid: draw $a a'$, $b b'$, $c c'$, $d d'$,

$e\,e'$ perpendicular to $x\,y$: join $a'\,v'$, $b'\,v'$, $c'\,v'$, $d'\,v'$, $e'\,v'$; this will complete the elevation of the pyramid. (See Fig. 73.)

(1.) To construct the elevation of a section made by a vertical plane whose trace $m\,n$ makes an angle β with $x\,y$, let this trace cut $d\,e$, $e\,v$, $a\,v$, $a\,b$ in the points o, p, q, r respectively. Then o and r being points in the horizontal plane, their elevations will be in the ground line; draw $o\,o'$ and $r\,r'$ perpendicular to $x\,y$; o' and r' (Chap. I. 26) will be the elevations of o and q: also p and q are the plans of points in the straight lines whose elevations are

Fig. 72.

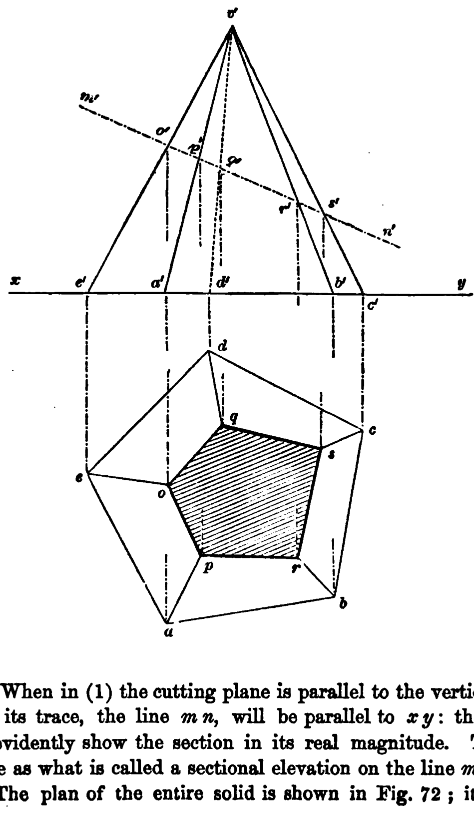


$a'\,v'$ and $b'\,v'$; if therefore from p and q straight lines be drawn perpendicular to $x\,y$ and meeting $a'\,v'$ and $b'\,v'$ in p' and q': p' and q' will be the elevations corresponding to p and q. Join $o'\,p'$, $p'\,q'$, $q'\,r'$: the figure $o'\,p'\,q'\,r'$ will be the elevation required.

(2.) Let the section be made by a plane perpendicular to the vertical plane: its trace $m'\,n'$ making an angle δ with $x\,y$

(Fig. 73), it is required to construct the plan of this section. Let the trace $m' n'$ meet the elevations $e' v'$, $a' v'$, $d' v'$, $b' v'$, $c' v'$ in the points o', p', q', r', s' respectively. The plans o, p, q, r, s will be found by drawing $o' o$, $p' p$, $q' q$, $r' r$ and $s' s$ perpendicular to $x y$ (Chap. I. 29), and meeting $e v$, $a v$, $d v$, $b v$, $c v$ in o, p, q, r, s respectively. The figure $o p r s q$ will be the plan of the section,

Fig. 73.



NOTE.—When in (1) the cutting plane is parallel to the vertical plane of projection, its trace, the line $m n$, will be parallel to $x y$: the elevation will then evidently show the section in its real magnitude. This would be the same as what is called a sectional elevation on the line $m n$.

Obs.—The plan of the entire solid is shown in Fig. 72; its elevation in Fig. 73.

PROBLEM VI.

To construct the projections of a prism; having given the inclination of the plane of its base, and that of one edge.

Let the prism be a hexagonal one; its base being the regular hexagon, A B C D E F (Fig. 4, Pl. I.), assume the plane P Q R, containing the base, to be perpendicular to the vertical plane of projection: the plan $a\,b\,c\,d\,e\,f$ of the base A B C D E F, and its elevation $a'\,b'\,f'\,c'\,e'\,d'$, may be found at once by Prob. XVII., Chap. III. The construction is omitted in the figure to avoid rendering it more complex.

From a', b', c', d', e', f' draw $a'a'_{,}$, $b'\,b'_{,}$, &c. &c., perpendicular to Q R: make $a'\,a'_{,}$ equal to the height of the prism; through $a'_{,}$, draw $a'_{,}\,b'_{,}f'_{,}\,c'_{,}\,e'_{,}\,d'_{,}$, parallel to Q R; this will complete the elevation of the prism.

To complete the plan it will be necessary to observe that the edges of the solid, being perpendicular to the plane P Q R, will be projected in straight lines perpendicular to P Q, the trace of that plane (Chap. I. 30). If therefore a straight line be drawn through a perpendicular to P Q, and one through $a'_{,}$, perpendicular to $x\,y$, the point $a_{,}$, in which these lines cut each other, will be the plan of the point whose elevation is $a'_{,}$. The plans of the other angular points of the upper end of the prism may be determined in a similar manner; and the plan of the solid will be completed, by joining these points. Since, however, the projections of parallel lines are parallel, after the point $a_{,}$ has been found, $b_{,}$ may be found by drawing through a a straight line parallel to $a\,b$ and meeting B b produced in $b_{,}$; a similar construction will give the other points.

To construct an elevation of the prism on a third plane of projection, at right angles to the other two.

Let $v\,u\,v'$ be this plane: turn it about the trace $u\,v'$ into coincidence with the vertical plane of projection, as shown in the figure.

Let A m, perpendicular to u v, meet u v in m: with u as a centre and radius u m, describe a circle cutting x y in m': through m' draw a straight line perpendicular to x y, and through a' and $a'_{,}$, straight lines parallel to u y, the points $a'_{,}$ and $a''_{,}$, in which these two lines respectively cut the former one, will evidently be the elevations of a and $a_{,}$, on this third plane. The other points may be found in the same way.

PROBLEM VII.

To construct the projections of a pyramid lying on one of its faces on the horizontal plane.

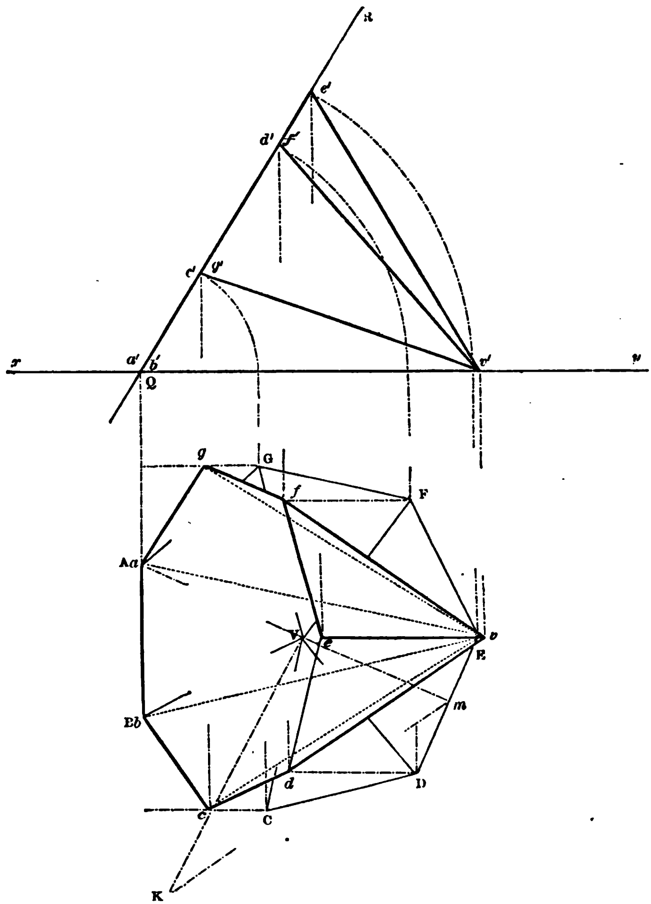
Suppose the pyramid to be a heptagonal one, with its axis parallel to the vertical plane of projection, and consequently the plane of its base perpendicular to that plane.

Let A B C D E F G, V (Fig. 74), be the plan of the figure standing on its base on the horizontal plane (Chap. IV. Prob. V.); A B being the side of the base in contact with the horizontal plane, and when produced meeting x y at right angles in the point Q: B Q will be the horizontal trace of the plane of the base.

Draw V m perpendicular to D E: V K perpendicular to V m, and equal to the perpendicular height of the pyramid: join m K: m K will be the slant height, and V m K the profile angle of any face and the base; and therefore the measure of the inclination of the base to the horizontal plane.

Through Q draw a straight line Q R, making with x y an angle R Q y, equal to V m K: Q R will be the vertical trace of the plane containing the base. The elevation of the base a' c' d' e' may now be found by Prob. XVII. Chap. III. Make a' v' equal to m K; join v' e', v' d', v c'; this will complete the elevation of the solid. The plans a, b, c, d, e, f, g and v, corresponding to the elevations a', b', c', d', e', f', g' and v', will be found by drawing through the elevations perpendiculars to x y (Chap. I. 29), and through A, B, C, D, E, F, G and V parallels to x y.

Fig. 74.



PROBLEM VIII.

To construct the projections of a rectangular parallelopiped, having given the inclination of the plane of one face, and the plan of one side of that face.

Let P Q (Pl. I. Fig. 5), be the horizontal trace of the given plane, α its angle of inclination: *a b* the given plan of one edge, *m* and *n* the lengths of the edges coterminous with that edge.

Find (Chap. II. Prob. XXIV.) Q R′ the vertical trace of the given plane: let the profile plane H R R′ employed in this construction pass through the point *b*, and turn it about its trace H R until it coincides with the horizontal plane (Chap. I. 35). Let H R R″ be the profile plane in this position: M R″ being its intersection with the given plane.

Draw *b* B perpendicular to H R: with centre M and radius M B describe a circle cutting H R in B′: B′ will be the position of the point whose plan is *b* when the plane P Q R′ has been turned about P Q into coincidence with the horizontal plane; join* *a* B′; *a* B′ will be the real magnitude of the edge given by its plan *a b* (Chap. I. 35). Draw *a* D perpendicular to *a* B′ and equal to *m*: the plan of D, which is the point *d*, will be found at once by the construction shown in the Figure. (Chap. I. 35). Complete the parallelogram *a b c d*; this will be the plan of the face in contact with the plane P Q R′. The edge perpendicular to this face and terminated in the point (*b*, *b*′) being perpendicular to the plane P Q R′ will be projected in R H which is at right angles to P Q (Chap. I. 30); whilst on the profile plane this edge will be represented in its real length B F equal to *n* and perpendicular to M R″: draw F *f* perpendicular to H R; *b f* will be the plan of the edge in question. Then, since the projections of parallel straight lines are parallel, the plan of the solid may be completed by drawing the parallelograms *a b f e*, *e a d h*, *g h d c*, *g f e h*, *b f g c*.

To construct the elevation. Since the lines whose plans are *a d* and *c d* lie in the plane P Q R′ their elevations will be determined by drawing vertical planes through *a d* and *c d*: the point *d*′ in which these elevations intersect will be the elevation of *d*: draw *c c*′ perpendicular to *x y* cutting *d*′ *c*′ in *c*′; *c*′ will be the elevation of *c*: complete the parallelogram *a*′ *b*′ *c*′ *d*′. The four edges perpendicular to the plane P Q R will have their elevations at right angles to Q R′ (Chap. I. 30): through *a*′ *b*′ *c*′ *d*′

* The line *a* B′ is omitted by mistake in Fig. 5, Pl. I.

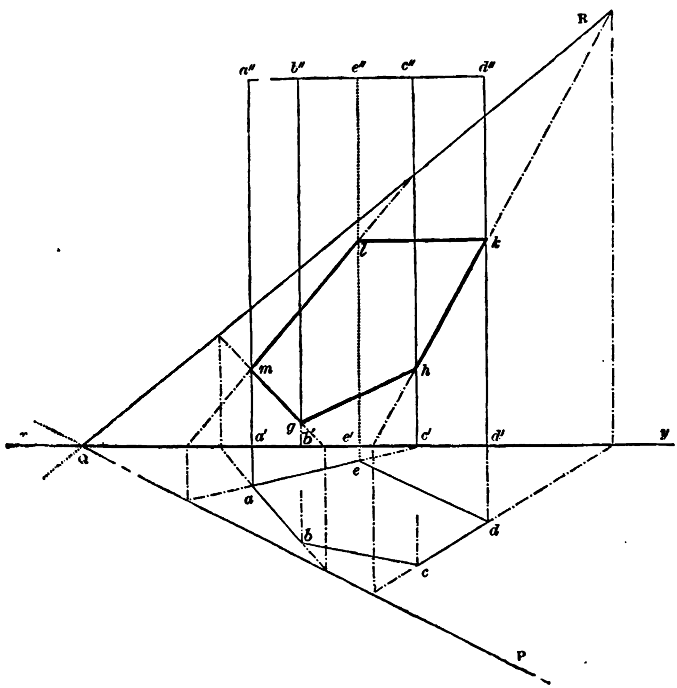
draw straight lines perpendicular to Q R′; and through *e*, *f*, *g*, *h* draw perpendiculars to *x y* meeting the perpendiculars to Q R′ in *e′*, *f′*, *g′*, *h′*; these points will be the elevations of *e*, *f*, *g*, *h* (Chap. I. 29). Join *e′ f′*, *f′ g′*, *g′ h′*, *h′ e′*, these lines will complete the elevation of the solid, and should be parallel respectively to *a′ b′*, *b′ c′*, *c′ d′*, *d′ a′*.

Problem IX.

To determine the projections of the intersection of a right prism by a given plane, the real magnitude of the section, and its development.

The section of a prism by a plane is a polygon whose sides are the lines in which the faces of the prism are cut by the plane,

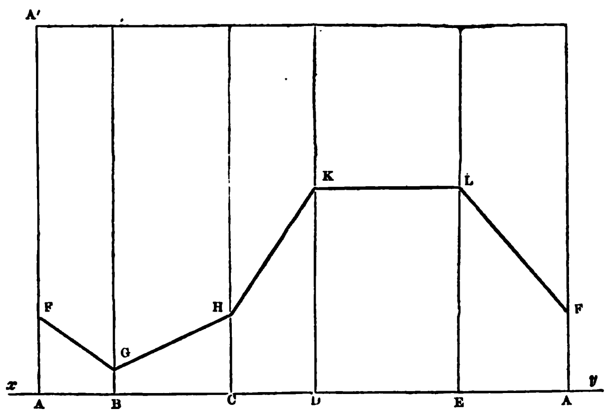
Fig. 75.



and whose angular points are the points in which the edges of the prism meet the plane.

When the prism is vertical the plan of the section will be the base, and the problem reduces itself to Prob. XXXV. Chap. II., viz., given the plan of a polygon, to find its elevation and real magnitude. The construction for determining the elevation by Prob. IX. Chap. II., is shown in Fig. 75, where *a b c d e* is the base of the prism and the plan of the section: *g h k l m* the elevation of the section. The real magnitude may be readily found by the above named problem.

Fig. 76.



The development of the prism will evidently be a rectangle whose base is equal to the sum of the sides of the figure *a b c d e*, and whose height is the height of the prism. Let A A′ be this rectangle, so that A B = *a b;* B C = *b c*; C D = *c d*; D E = *d e*; E A = *e a*. Through B, C, D, E draw perpendiculars to *x y*; make A F = *a′ m*; B G = *b′ g*; C H = *c′ h*; E L = *e′ l*: the line F G H K L F will be the development of the section (Fig. 76).

NOTE.—The intersection of a right cylinder by a given plane may be determined by a similar construction. This and some of the following problems may be simplified by assuming the trace P Q perpendicular to *x y*.

Problem X.

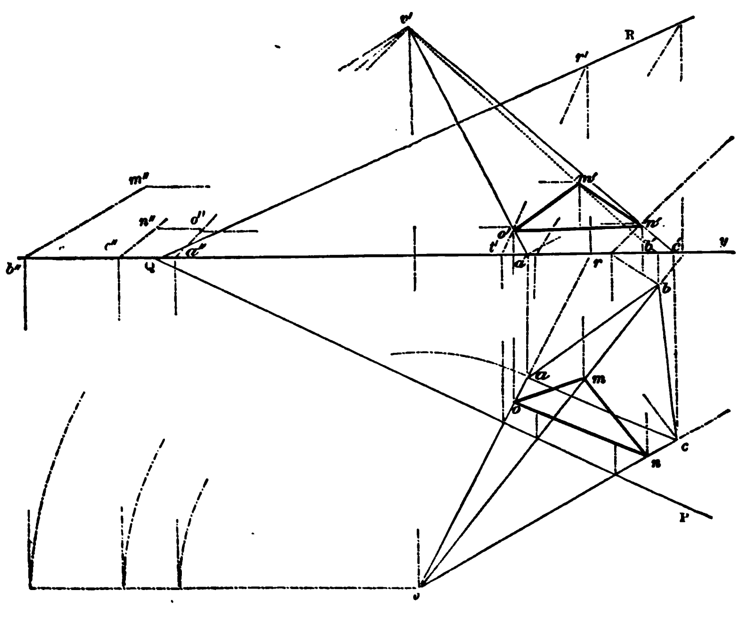
To find the projections of the intersection of a pyramid by a given plane: to determine the real magnitude of the section and its development.

The section of a pyramid by a plane is a polygon whose sides are the intersections of the faces by the plane, and whose angular points are the points in which the edges of the pyramid meet the plane.

Let P Q R (Fig. 77) be the given plane; *a b c* the base of the pyramid, which is supposed to stand on its base on the horizontal plane, (v, v') being the vertex. Complete the projections of the solid, as shown in the figure.

The angle of the section situated on the edge ($v\,a$, $v'\,a'$) will be

Fig. 77.

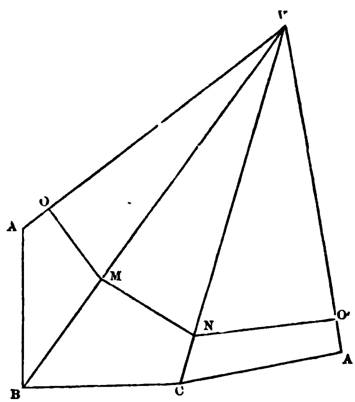


the point (o, o'), in which that edge meets $(v\,r, t'r')$, the intersection of its vertical projecting plane with the plane P Q R. The remaining angular points may be determined in the same manner; and by joining these points the section $(m\,n\,o,\ m'n'o')$ will be completed (Prob. IX. Chap. II.).

The real magnitude of the section will be found at once by Prob. XXXV. Chap. II.

The development will evidently consist of as many triangles as there are sides to the base. These triangles will have their bases equal to the sides of the base of the pyramid; the vertex V (Fig. 78) common, and the sides V A, V B, V C, equal respectively

Fig. 78.



to the edges $(v\,a,\ v'a')$, $(v\,b,\ v'b')$, $(v\,c,\ v'c')$. The lengths of these edges will be found, by turning them about the vertical line $v\,v'$ until they are parallel to the vertical plane of projection, as shown in Fig. 77, to be $v'a''$, $v'b''$, $v'c''$. Make $\mathrm{V\,A} = v'a''$; $\mathrm{V\,B} = v'b''$; $\mathrm{A\,B} = a\,b$; $\mathrm{B\,C} = b\,c$; $\mathrm{V\,C} = v'c''$; $\mathrm{C\,A} = c\,a$; $\mathrm{V\,A} = v'a''$. This will be the development of the pyramid. Draw $o'o''$, $m'm''$, $n'n''$ parallel to $x\,y$; make $\mathrm{V\,O} = v'o''$; $\mathrm{V\,M} = v'm''$; $\mathrm{V\,N} = v'n''$; $\mathrm{V\,O} = v'o''$; join O M, M N, N O. This will be the development of the perimeter of the section.

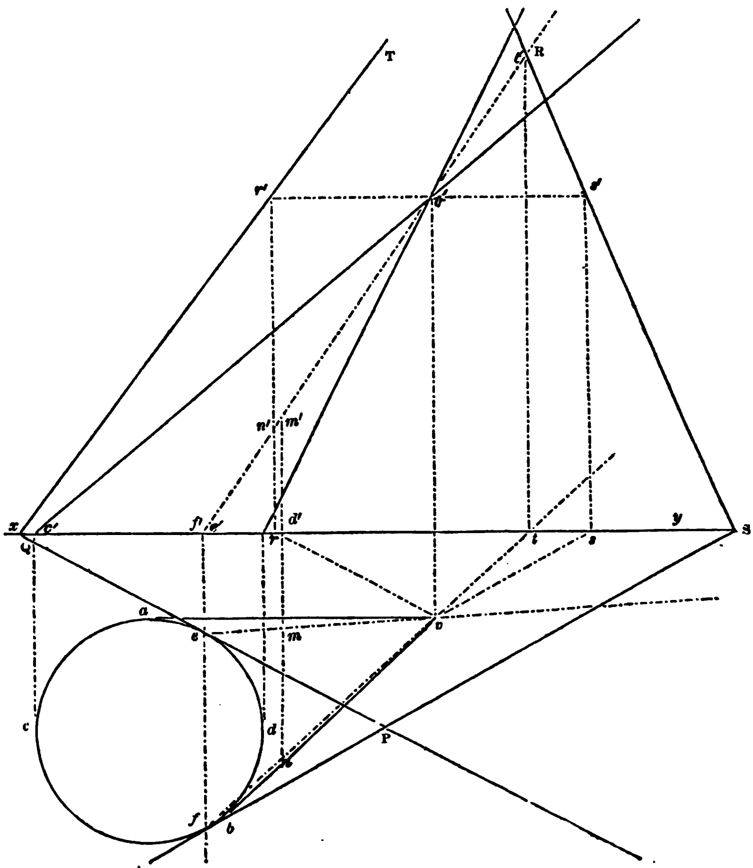
NOTE.—The section of a right cone may be determined in a similar manner.

Problem XI.

To draw a tangent plane to a given cone through a given point on its surface.

Let $a\,c\,b\,d$ (Fig. 79) be the horizontal trace of the cone; (v, v') its vertex.

Fig. 79.



To determine the limits within which the projections of all generatrices must lie: draw in the horizontal plane the tangents $v\,a$, $v\,b$; the plans of all generatrices will lie between $v\,a$ and $v\,b$. Draw $c\,c'$ and $d\,d'$ tangents perpendicular to $x\,y$; join $v'c'$ and $v'd'$. The elevations of all generatrices will be between $v'c'$ and $v'd'$.

Next, to find the points on the surface corresponding to a given projection: let m' be the given projection. Draw the straight line $v'm'$, cutting $x\,y$ in e'; draw $e'e$ perpendicular to $x\,y$; $v'e'$ and $e'e$ are the traces of a plane perpendicular to the vertical plane of projection, containing the required points, and passing through the vertex of the cone. This plane cuts the surface of the cone in two generatrices (Chap. III.); and since its horizontal trace meets $a\,c\,b\,d$ in e and f, it is manifest that the plans of the generatrices are the straight lines $v\,e$ and $v\,f$. Draw $m'm$ perpendicular to $x\,y$, meeting $v\,e$ in m and $v\,f$ in n; m and n will be the plans of the points whose elevation is m' or n'.

To draw a tangent plane through the point (n, n'), it must be remembered that this plane will contain the generatrix ($v\,f$, $v'f'$), and touch the cone throughout the entire extent of that line (Chap. III.). Draw P S, touching $a\,c\,b\,d$ in f; find (Chap. II. Prob. III.) t' the vertical trace of (vf, $v'f'$); through t' draw R S: P S R will be the plane required.

In the same way a second tangent plane may be determined, as P Q T, containing the generatrix $v\,e$, $v'e'$.

If through any point in the generatrix a straight line ($v\,r$, $v'r'$) be drawn parallel to the trace P Q, the vertical trace of this line will evidently be a point in the vertical trace of the plane. This sometimes affords a means of determining the vertical trace of the plane, when that of the generatrix falls beyond the limits of the drawing. For example, the vertical trace Q T was found by means of r', the trace of (vr, $v'r'$), because that of ($v\,e$, $v'e'$) could not be conveniently determined.

Problem XII.

To draw a tangent plane to a given cone through a given point without it.

Every plane touching the cone passes through the vertex (Chap. III. 15), and has its horizontal trace a tangent to the base of the cone.

The problem may therefore be solved as follows;—

Join the given point with the vertex of the cone; determine the traces of the line joining these points (Chap. II. Prob. III.); through its horizontal trace draw tangents to the base of the cone. These tangents will be the horizontal traces of two planes, fulfilling the conditions of the problem. Their vertical traces will be the straight lines drawn through the vertical trace of the said line, and the points in which the horizontal traces meet the ground line.

Problem XIII.

To draw a tangent plane to a given cone parallel to a given straight line.

Since the plane must pass through the vertex of the cone, the problem may be solved by the following construction:—

Through the vertex of the cone draw a straight line parallel to the given line (Chap. II. Prob. II.). Determine the traces of this line (Chap. II. Prob. III.). Through its horizontal trace draw a tangent to the base of the cone. This will be the horizontal trace of the required plane. Through the point in which this trace meets the ground line and the vertical trace of the line passing through the vertex, draw a straight line; this will be the vertical trace of the plane.

Exercises.

1. Define the terms cylinder, pyramid, cube; and explain what is meant by the profile angle of two planes.

2. Draw the plan and side elevation of a pentagonal prism, three inches long, laid on one face. Put flat shades of Indian ink on the faces, making them darker in proportion as they recede from the horizontal in the plan, and vertical in the elevation. Side of base $1\frac{1}{4}$ inches.

3. Find the intersection of a plane inclined at 40° to the horizon with a cylinder, having for its bases circles of 1 inch in diameter, and standing in a vertical position.

4. A hexagonal right prism, 9 feet long, each edge of the base measuring 2 feet, rests with one face on the horizontal plane. Draw its plan, and an elevation on a plane, making an angle of 27° with the axis of the prism. Scale $\frac{1}{24}$.

5. A tetrahedron, with an edge of 2 inches, rests on one of its faces. Draw its plan, and a sectional elevation on any plane not passing through the apex, and not parallel to an edge of the base.

6. Draw the plan of the frustum of a hexagonal pyramid, the side of the base being $1\frac{1}{2}$ in., the side of the top 1 in., and the height $2\frac{1}{2}$ in., one side of the base being horizontal, and the plane of the base inclined 27° to the horizon.

7. A rectangular parallelopiped, 12 ft. by 10 ft. by 8 ft., lies on its base on a plane inclined to the horizon at an angle of 20°, the horizontal trace of the plane making, with the line of level, an angle of 25°. The projection of the longest side of the base (=12 ft.) is inclined to the trace of the plane at an angle of 10°. Construct the horizontal and vertical projections of the body. Scale $\frac{1}{60}$.

8. The dimensions of a cross are as follows:—

Pedestal 3′ × 3′ and 1′ in height; shaft 1′ × 1′ and 7′ in height, 1′3″ of which is clear above the arms; arms 1′ × 1′ in section, and 1′6″ long. Draw a front elevation of the cross, its plan, and a section on a line crossing the arms diagonally. Scale $\frac{1}{24}$.

9. Construct a pentagon A B C D E on a side A B equal to 1·3 in.; and considering this the base of a prism 2 in. high, standing on a horizontal plane, draw a vertical section on a line passing through A B.

10. Draw an elevation of the prism in Ex. 9 on a plane, making an angle of 5° with one of the faces.

11. A prism, 3 inches long, having for its ends regular hexagons of 1 inch side, is laid on one of its faces. Draw the plan; and an elevation on a line, making an angle of 30° with the side of the plan.

12. Draw the plan of a pentagonal pyramid, resting upon the plane of its base, each side of the base being 2 inches long, and the height 3 inches. Draw an elevation of the pyramid on a vertical plane parallel to one of the sides of the base.

13. Draw the plan of a frustum of a square pyramid: side of large end 2 inches; side of small end 1 inch; height 3 inches. Draw the elevation on a line making an angle of 40° with one side of the plan of the base.

14. Draw the plan of the prism in Ex. 9, resting on one of the edges of its base; the base being inclined to the horizon at an angle of 15°.

15. Draw the plan of an octahedron, of 2·5 in. edge, lying on one of its faces; and make a section and elevation on any line not parallel to a side of the plan.

16. A prism, 2 inches high, the base of which is a hexagon 1 in. side, stands on a plane inclined at an angle of 40° to the horizon, and has one diagonal of its base inclined at 25° to the horizon. Draw its plan.

17. The face of a cube whose edge measures 2 inches, is inclined 50° to the horizon, and one of the diagonals of this face 25°; project the cube. Draw a horizontal contour $\frac{1}{2}$ inch vertically below the highest point of the cube, and make an elevation of the cube on a vertical plane, parallel to any one of its diagonals.

18. A right prism, each edge of which measures 15 feet, stands with its base, which is a regular pentagon with a side of 8 feet, upon a horizontal plane; draw its plan, and also its elevation on a vertical plane making an angle of 20° with any side of the base. Scale 5 feet to an inch.

19. Supposing the prism in Ex. 10 to be oblique instead of right, its axis making an angle of 40° with the plane of the base, draw its plan and elevation as before.

20. A plane is inclined at $46\frac{1}{4}°$ to the horizon; on it is placed a tetrahedron of 3 inches edge, and having one side of its base inclined at 25°. Draw the plan and elevation, and contour the former at vertical intervals of ·25 inches.

21. Construct the plan of a hexagonal pyramid resting on one of its faces, its height being 35 feet, and each side of its base 15 feet. Scale 10 feet to an inch.

22. A dodecahedron, formed by two hexagonal pyramids joined at their bases, is laid on a face. Draw its plan; also a sectional elevation on a line cutting the junction of the pyramids at an angle of 45°. Height of pyramid $2\frac{1}{2}$ inches, side of base $1\frac{1}{4}$ inch.

23. Draw a plan, a cross section, and a front elevation of a cottage, 25 feet long, 15 feet wide (interior dimensions), walls 1·5 feet thick, a door in the centre of one side, 4 feet wide and 7 feet high, and a window on either side of it 3 feet wide, 4 feet high, and 4 feet above the ground. The walls of the cottage are 10 feet high, the roof has two gables, and the ridge is 16·5 feet above the floor; the eaves project 9 inches. Scale $\frac{1}{84}$.

24. A cubical block of masonry 20 feet high, having a frontage of 17 feet, and a depth of 13 feet, is perforated longitudinally and transversely by semicircular arches, springing at the same height, viz., 10 feet 6 inches, and intersecting each other. The piers are, in plan, 4 feet 3 inches square.

Draw the plan of the structure, its front elevation, and a transverse section through the corner of the arch. Scale $\frac{1}{96}$.

25. Construct the plan of a cube 2·5 inches side, having two of its adjacent edges inclined 30° and 40° to the horizon. Show the intersection with the cube of a horizontal plane 1 inch below its highest point.

CHAP. V.

ON SHADOWS.

1. *Def.*—Every deprivation of direct light produces on the surface of bodies an obscurity, more or less intense, which is termed a SHADOW.

The Theory of Shadows is based upon the principle that light proceeds in right lines. The determination of shadows comprises two entirely distinct parts; one of these consists in finding the outlines of shadows, the other relates to the depth of the tint to be assigned to each particular portion of the surfaces that receive the shadows. The former of these, as an application of Descriptive Geometry, will alone be treated of here.

2. *Def.*—A RAY OF LIGHT is the term applied to that portion of light which may be looked upon as coincident with a straight line drawn from any point of the luminous body to a point of the object illuminated.

3. If the luminous body be at a very great distance from the body illuminated, as, for example, the sun from the earth, the rays of light may, for all practical purposes, be regarded as parallel to each other. In the present chapter, all objects will be supposed to be illuminated by direct solar light. The direction of the rays will be given by their inclination to one or more given planes.

4. *Def.*—Shadows are divided into two kinds, SHADOWS PROPER, and SHADOWS CAST. A SHADOW PROPER is that which takes place on that portion of the opaque body which is turned away from the light. A SHADOW CAST is that which is produced on a surface by the opaque body intercepting those rays of light which would otherwise illuminate it.

5. *Def.* — THE LINE OF SEPARATION OF LIGHT AND SHADOW is the line which separates the illuminated portion of a body from that which is not so; it is determined by the contact of luminous rays with the surface of the body, and is therefore the apparent outline of a body to a spectator whose eye is situated at a point infinitely distant from the body, on a straight line drawn from the body, parallel to the direction of the luminous rays.

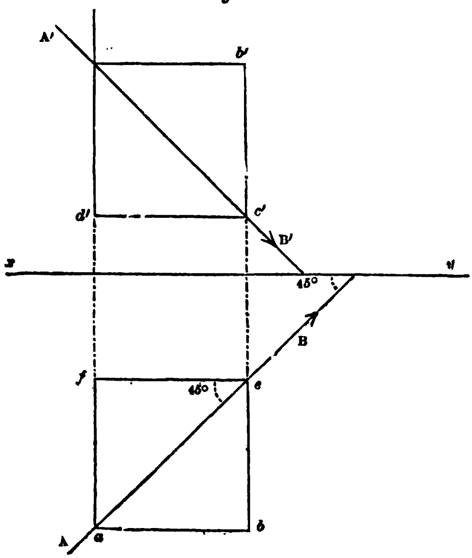
To determine the outline of the shadow cast by an opaque body upon any surface, it will be necessary to find the points in which the rays tangential to the body meet the given surface; the straight line joining these points will be the outline required.

6. The line of separation on a Polyhedron is composed of those edges of the solid which are common to two faces, one of which is struck by the rays of light, the other not. For the face of a polyhedron is either wholly illuminated, or, when other faces are interposed between its plane and the source of light, wholly obscure. It will therefore be evident that the line of separation on a polyhedron must be composed of those edges which separate the light faces from the dark ones. The question of determining this line thus depends upon finding those edges. Now, two adjacent edges of a polyhedron will manifestly be, one illuminated, the other not, when the plane drawn through their common edge, parallel to the direction of the luminous rays, leaves both of these faces on the same side of it; for the rays will reach the face in front, but will not reach the face behind, because they are prevented from so doing by the first face, and others contiguous to it. If, on the contrary, the plane parallel to the rays of light enters the dihedral angle of two faces, these faces will be both illuminated when the angle is turned towards the luminous source; both deprived of light when the angle is turned away from that source. It will easily be perceived that the plane drawn through an edge parallel to the rays of light, leaves on one side the two faces to which that edge is common, when the trace of that plane on one of the planes of projection is exterior to the angle formed by the traces of these planes on the same plane.

In the following problems, when nothing is stated to the contrary, the direction of the rays of light will be assumed parallel to the diagonal of a cube whose opposite faces are parallel to the planes of projection. The projections of the rays will then make angles of 45° with the ground-line.

Let *a b e f* (Fig. 80) be the plan, *a′ b′ c′ d′* the elevation of a

Fig. 80.



cube so situated; then A′ B′ will be the elevation, and A B the plan of a ray of light coincident with the diagonal; and it is clear that A B and A′ B′ are inclined at angles of 45° to *x y*.

Problem A.

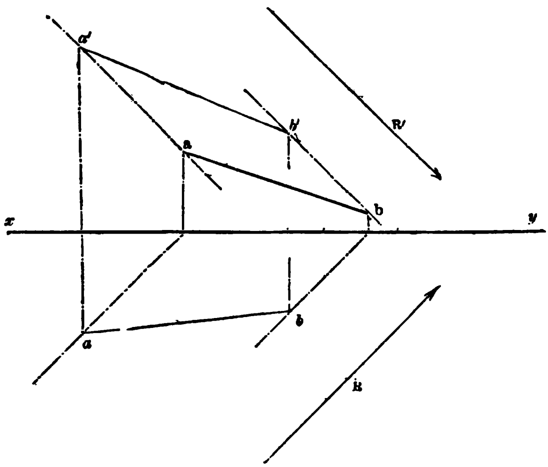
To determine the shadow cast by a physical point upon the vertical plane of projection.

Let R and R′ (Fig. 81) be the projections of a ray of light, (*a*, *a′*) the given point.

Then, from what has been stated in the introductory remarks, it will be perceived that the shadow cast by the point upon the vertical plane, will be the vertical trace of the ray of light passing through the point.

Through (a, a') draw a straight line parallel to (R, R′) (Chap. I., Prob. II.); find the vertical trace, a, of this line (Chap. II., Prob. III.); the point a will be the shadow required.

Fig. 81.



Similarly, the shadow cast upon the horizontal plane may be found.

Cor. Let $(a\,b,\ a'\,b')$ be a straight line; then, since the shadow of a straight line upon a plane is a straight line, if the shadows of (a, a') and (b, b'), determined as above, be a and b, the straight line a b will be the shadow cast by the line $(a\,b,\ a'\,b')$ upcn the vertical plane.

Its shadow upon the horizontal plane might be determined in a similar manner.

NOTE.—If a straight line be parallel to a plane, its shadow upon that plane will be equal and parallel to the line itself. *Euc.* I. 33.

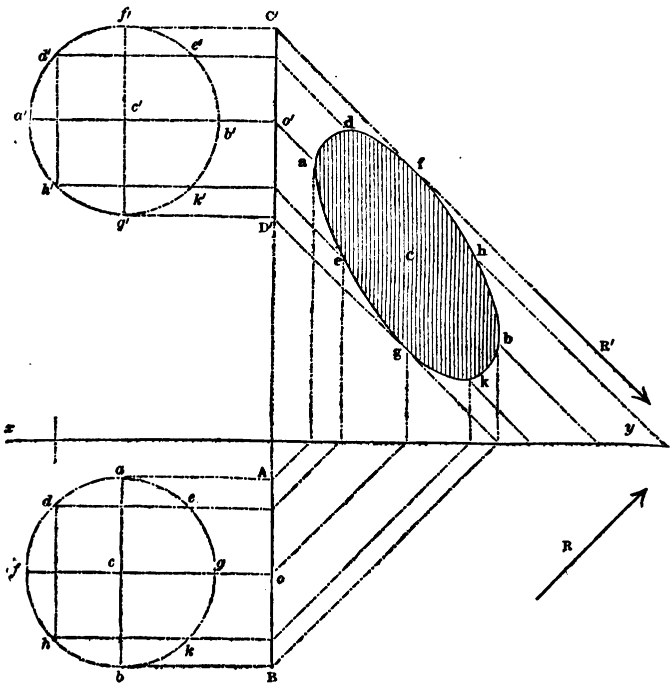
Problem B.

To determine the shadow cast by a circle upon the vertical plane of projection.

(1.) Let the plane of the circle be parallel to the vertical plane; the shadow will then be a circle equal to the original one. Find the shadow of the centre by Prob. A: with this point as a centre, and a radius equal to that of the given circle, describe a circle, this will be the shadow required.

(2.) Let the circle be perpendicular to the planes of projection: its projections (which will be straight lines equal to its diameter,

Fig. 82.



and perpendicular to $x\,y$) being A B and C′ D′. Describe the circle in both planes as $a\,f\,b\,g$ and $a'\,f'\,b'\,g'$: divide these circles into equal parts and project the points of division upon A B and C′ D′ respectively. Let the shadows of these points found by Prob. A, be a, d, f, h, b, k, g e, then the curve a d f h b k g e will be the shadow sought, c being the shadow of the centre.

Problem C.

To determine the line of separation on a hexagonal pyramid and the shadows cast by it on the planes of projection.

Let (v, $a\,b\,c\,d\,e\,f$, v', $a'\,b'\,c'\,d'\,e'\,f'$), (Pl. II. Fig. 1) be the pyramid standing on its base on the horizontal plane, (v, v') being the vertex.

It is evident (6) that the faces ($v\,a\,f$, $v'\,a'\,f'$), ($v\,e\,f$, $v'\,e'\,f'$) and ($v\,e\,d$, $v'\,e'\,d'$) are the only ones that are illuminated; ($v\,a$, $v'\,a'$) and ($v\,d$, $v'\,d'$) will therefore constitute the line of separation; the elevation $v'\,a'$ not being visible. The problem will be solved by tracing the shadow of the lines ($a\,v$, $a'\,v'$) and ($d\,v$, $d'\,v'$). To effect this find (Prob. A) V the shadow of the vertex on the horizontal plane: join a V, d V; this will give the outline of the shadow cast upon the horizontal plane. If the pyramid be sufficiently near the vertical plane, which will be shown by the point V falling above $x\,y$, a portion of the shadow will fall upon that plane. When such is the case determine V′, the shadow of the vertex on the vertical plane (Prob. A): let a V and d V cut $x\,y$ in $\underline{m}$ and $\underline{n}$ respectively: join V′ $\underline{m}$ and V′ $\underline{n}$; V′ $\underline{m}\,\underline{n}$ will be the portion of the shadow falling upon the vertical plane.

(Fig. 2, Pl. II.) shows the shadow of a hexagonal prism, obtained in a similar manner, and consequently needing no explanation.

PROBLEM D.

To determine the line of separation on a right cylinder with circular base, and the shadow cast by it on the planes of projection.

Let the cylinder stand upon its base upon the horizontal plane. The determination of the line of separation reduces itself to drawing two tangents to the base parallel to the direction of the light, and to drawing through the points of contact two generatrices of the cylinder.

Let $a\,b\,c\,d\,e$ and $a'\,c'\,c''\,a''$ be the plan and elevation of the solid (Pl. II. Fig. 3), draw $e\,\underline{m}$ and $b\,\underline{n}$ parallel to R the plan of a ray of light and touching the circle $a\,b\,c\,d$, in b and e; draw $e\,e''$ and $b\,d''$ perpendicular to $x\,y$; $e'\,e''$ and $d'\,d''$ will be the line of separation, $e'\,e''$ being invisible. Let $e\,\underline{m}$ and $b\,\underline{n}$ meet $x\,y$ in $\underline{m}$ and $\underline{n}$; find (Prob. A,) $\underline{e}, \underline{f}', \underline{d}', \underline{c}', \underline{g}'$, the shadows of points, in the upper end, on the vertical plane. The curve $\underline{e}\,\underline{f}'\,\underline{d}'\,\underline{c}'\,\underline{g}'$ and the lines $\underline{e}\,\underline{m}$, $\underline{g}'\,\underline{n}$ will complete the shadow thrown on that plane. Had the shadow fallen wholly on the horizontal plane, it would have determined by describing a circle with the shadow of the centre for a centre and touching $h\,\underline{m}$ and $h\,\underline{n}$ as shown in the diagram.

PROBLEM E.

To determine the line of separation on a right cone, and the shadows cast by it on the planes of projection.

1. Let the cone $(v, a\,b\,c\,d\,e, v', a'\,c')$ (Pl. II. Fig. 4), stand on its base upon the horizontal plane. Determine, by Prob. A, v the shadow of the vertex; from v draw $\mathrm{v}\,b$ and $\mathrm{v}\,e$, touching the base in b and e; this will give the outline of the shadow cast upon the horizontal plane. Join the points of contact e and b with v; the lines $(v\,e, v'\,e')$ and $(v\,b, v'\,b')$ will be the line of separation, $v'\,b'$ alone being visible in the elevation.

When a portion of the shadow falls upon the vertical plane it may be determined as in Problem C. (Fig. 4.)

2. Let the cone be inverted, v, $a\,b\,c\,e$ and $a'\,v'\,d'$ (Pl. II. Fig. 5) being its projections. Determine $\underline{o}$ the shadow of the centre (Prob. A); with $\underline{o}$ as a centre and a radius equal to that of the base of the cone, describe the circle $\underline{m}\,\underline{p}\,\underline{n}$; from o draw $o\,\underline{m}$ and $o\,\underline{n}$, touching this circle in $\underline{m}$ and $\underline{n}$; this will complete the outline of the shadow cast upon the horizontal plane. Again from o draw $o\,b$ and $o\,e$ parallel to $\underline{o}\,\underline{m}$ and $\underline{o}\,\underline{n}$; the lines $(v\,e, v'\,e')$ and $(v\,b, v'\,b')$ will form the line of separation; the elevation $v'\,b'$ being alone visible.

When a portion of the shadow falls upon the vertical plane, that part may be determined by finding the shadows of equidistant points in the base and tracing an elliptical curve through them, in a manner similar to that in Problem B. (Fig. 5.)

Problem F.

NUMERICAL EXAMPLE.

To draw the shadow cast upon the horizontal plane by a solid formed of two pentagonal pyramids joined at their bases. The rays of light making an angle of 50° with the horizontal and of 15° with the vertical planes of projection; edge of base ·895 inches; perpendicular height of each pyramid 1·25 inches; one edge of base making an angle of 34° 30′ with vertical plane.

Let v_1, $a\,b\,c\,d\,e$ (Fig. 6, Pl. II.) be the plan, and $v'_1\,a'\,v'_2\,c'$ the elevation of the solid determined as shown in Chap. IV.

The line of separation found by the principle enunciated in Chap. V. 5, will consist of the lines $(v_1\,e, v'_1\,e')$, $(v_1\,b, v'_1\,b')$, $(v_2\,e, v'_2\,e')$, and $(v_2\,b, v'_2\,b')$. Through (v_2, v'_2) draw a straight line making an angle of 50° with the horizontal plane, and an angle of 15° with the vertical (Chap. II. Prob. XV.): find the horizontal trace of this line (Chap. II. Prob. III.): let it be v_2; v_2 will be the shadow of the point (v_2, v'_2) (Chap. V. 5). The shadows $\underline{e}$ and $\underline{b}$ of the points (e, e') and (b, b') will be found in the

same way: join $\underline{e}\,v_2$, $\underline{b}\,v_2$; these lines will be shadows of the lines $(v_2\,e, v'_2\,e')$ and $(v_2\,b, v'_2\,b')$: the shadows of $(v_1\,e, v'_1\,e')$ and $(v_1\,b, v'_1\,b')$ will be found by joining $v_1\,\underline{e}$ and $v_1\,\underline{b}$, because the point $(v_1\,v'_1)$ is in the horizontal plane.

EXERCISES.

1. Draw the elevation of a cube of $2\frac{1}{2}$ inches edge, lying on one of its faces, on a plane which makes an angle of 25° with another face of the cube: project also the shadow thrown by the cube on the plane on which it lies, the rays of light being parallel to the plane on which the elevation is taken, and making angles of 50° with the horizon.

2. Follow the instructions given in the preceding question, only let the cube now rest on one of its edges, and a face containing that edge be inclined 25° to the horizon.

3. Project the shadow which the frustum in Ex. 6, Chap. IV. will throw upon the horizontal plane passing through the lower edge of the base, the plan of the rays being parallel to that edge, and inclined 37° to the horizon.

4. A pyramid with a pentagonal base, B C D E F, of $1\frac{1}{2}$ inch side, resting on a horizontal plane, has its vertex A 2 inches perpendicularly over a point within the base, ·88 inch from the angles at B and C. Draw the plan, and a section on a line joining the angles B and D.

5. Supposing the rays of light to be parallel, and to form with the horizon an angle of 65°, give the lines of the shadow on the horizontal plane on which the pyramid rests, when one of the edges of the base is inclined to the rays of light at an angle of 45°.

6. Project the shadow which the oblique prism in Ex. 19, Chap. IV. would cast upon the horizontal plane upon which it stands, the rays of light making an angle of 50° with the horizon, and being parallel to the axis of the prism.

7. A square pyramid, of 2 inches edge, is placed with the vertex downwards and raised 1 inch above the paper. Draw its plan

and its shadow when the rays are inclined at an angle of 30° to the horizon.

8. Draw the plan of an octahedron, or double square pyramid, with the line joining the vertices in a vertical position; and the elevation on a vertical plane, the plan of which makes an angle of 35° with a side of the plan of the octahedron. Also the shadow thrown by a ray, making an angle of 40° with the horizon and 36° with side of plan. Put a flat tint on the shadow and on the faces in shade, darkening those most shaded. Edge of solid 2 inches.

9. Draw the plan of a tetrahedron, the edge being 2½ inches, and a sectional elevation of the solid on any vertical plane not passing through the vertex. Determine the shadow thrown by the solid on the plane of its base, the rays of light being parallel to one edge of the base and inclined at an angle of 30° to the horizon.

10. Project the shadow that would be thrown by the pyramid in Ex. 21, Chap. IV. upon the horizontal plane upon which it rests, the parallel rays of light making an angle of 54° with the horizon, and a horizontal angle of 36° with the axis of the pyramid.

11. Draw the shadow which would be cast by the frustum, Ex. 6, Chap. IV., on the horizontal plane, when the sun's rays make with the horizon an angle of 49°, their plane being parallel to the diagonal of the base of the solid.

12. Draw the shadow which the prism in Ex. 9, Chap. IV. would throw on the plane on which it stands, supposing the rays of light to fall on one of the faces in a direction inclined at 45° both with its vertical and with its horizontal boundary.

13. Project the shadow that would be thrown by the pyramid upon the horizontal plane upon which it stands, the parallel rays of light making an angle of 40° with the horizon. Ex. 12. Ch. IV.

14. Draw the plan of a cylinder 3 inches long resting on its side, and project the shadow when the ray of light makes an angle of 40° with the horizon, and its plan an angle of 30° with the side of the plan of the cylinder. Diameter of end 2 inches.

CHAP. VI.

ISOMETRIC PROJECTION.

THIS method of projection, based upon that of a cube situated with its diagonal perpendicular to the plane of projection, affords a means of representing objects in a manner somewhat resembling what is called a "bird's-eye view." Its adaptability to this purpose was first pointed out by Professor Farish, of Cambridge.

The term isometrical is applied to it because the projections of all straight lines parallel to any edge of the cube may be measured from the same scale. This is evident, because such lines are all inclined at the same angle to the plane of projection; and, consequently, the projections of any two lines will have the same ratio as the lines themselves.

This kind of projection is peculiarly suitable to the delineation of objects whose bounding surfaces lie in three planes which form a right trihedral angle.

PROBLEM.

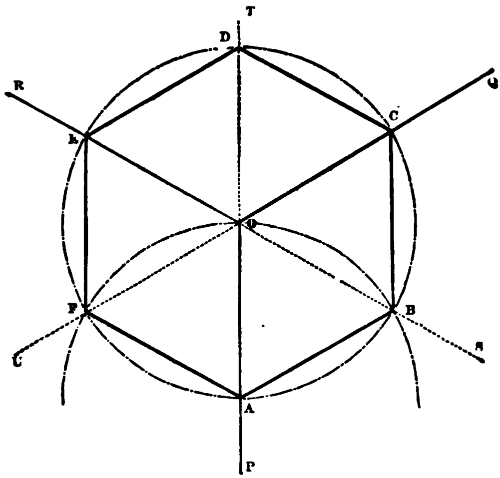
To determine the projection of a cube with its diagonal perpendicular to the plane of projection.

Let O (Fig. 83) be the projection of the diagonal which is perpendicular to the plane of projection.

Then the three edges which meet the lower extremity of this diagonal will be projected in three straight lines passing through O; and making angles of 120° with each other. For these lines make equal angles with each other, and are inclined at the

same angle to the plane of projection. Let these projections be O S, O T, O U.

Fig. 83.



To obtain the projections of the three edges coterminous with the upper extremity of the diagonal, it must be observed: (1.) that O is a point in each of them; (2.) that their projecting planes are the same as those of O S, O T, O U; these projections will, therefore, be found by producing S O, T O, U O, as O R, O P, O Q.

It is thus seen that the six edges of the cube coterminous with the diagonal are projected in six straight lines radiating from O; and that each of the angles S O P, S O Q, Q O T, T O R, R O U, U O P, is equal to one-sixth of four right angles, or 60°: also, that the projections of an upper edge and a lower one are in the same straight line.

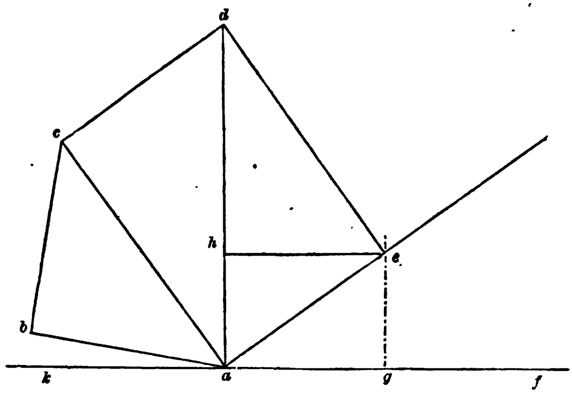
It remains to determine the magnitude of these projections, which, as before stated, will be equal to one another.

Let *a b* (Fig. 84) be the edge of the cube: draw *b c* at right angles to *a b* and equal to it: join *a c*: draw *c d* at right angles to *a c*, and equal to *a b*: *a d* will be the diagonal of the cube:

complete the rectangle *a c d e*, this will be the diagonal plane of the cube; and will be at right angles to the plane of projection when *a d* is. (*Euc.* XI. 19.)

Draw *k f*, through *a*, perpendicular to *a d*: draw *e g* perpendicular to *k f*: *a g* will manifestly be the projection of *a e*.

Fig. 84.



Referring again to Fig. 83, with centre O and radius O A, equal to *a g*, describe a circle meeting the projections in A, B, C, D, E, F: O A, O B, O C, O D, O E, O F will be the projections of the six edges which meet the diagonal whose projection is O.

The projections of the remaining six edges must be the lines A B, B C, C D, D E, E F, F A; which are equal to one another and to O A, because the triangles A O B, B O C, &c., are equilateral.

Cor. 1. The outline of the projection of a cube is a regular hexagon inscribed in the circle whose radius is equal to the projection of an edge.

Cor. 2. The projection of each face is a rhombus, composed of two equilateral triangles, consequently its angles are 120°, 120°, and 60°, 60°.

Cor. 3. The ratio between a straight line and its isometric

projection may be thus determined. See Fig. 84. Let $n =$ the edge of the cube, draw $e\,h$ perpendicular to $a\,d$. Then since $a\,b = b\,c = n$; $a\,c = n\sqrt{2}$, and since $c\,d = n$; $a\,d = n\sqrt{3}$. (*Euc.* I. 47.)

Again (*Euc.* VI. 8):

$$a\,d : d\,e :: d\,e : d\,h \therefore d\,h = \frac{2\,n}{\sqrt{3}}$$

$$\text{also, } a\,d : a\,e :: a\,e : a\,h \therefore a\,h = \frac{n}{\sqrt{3}}$$

$$\text{and, } d\,h : e\,h :: e\,h : a\,h \therefore e\,h = \sqrt{\frac{2\,n}{\sqrt{3}} \times \frac{n}{\sqrt{3}}} = \frac{n\sqrt{2}}{\sqrt{3}}$$

but $a\,g = e\,h \therefore a\,g = \frac{n\sqrt{2}}{\sqrt{3}} = \frac{n}{3}\sqrt{6} = \cdot 8165\,n$, nearly.

The required ratio is, therefore, $3 : \sqrt{6}$ or $1 : \cdot 8165$ nearly.

Cor. 4. If θ be the inclination of the edge:

$$\text{Cos}\,\theta = \frac{a\,g}{a\,e} = \cdot 8165 \therefore \theta = 35^\circ\ 16'.$$

If ϕ be the inclination of a face:

$$\text{Sin}\,\phi = \text{Cos}\,\theta = \cdot 8165 \therefore \phi = 54^\circ\ 44'.$$

Def.—The Lines O P, O Q, O R, are called axes; and lines can be projected isometrically only when they are parallel to some one of the lines projected in O P, O Q, O R.

Exercise 1. To construct the isometrical projection of a scale of $\frac{1}{12}$.

Let $a\,e\,d$ (Pl. II. Fig. 7) be a right-angled triangle, constructed as in Fig. 84, the side $a\,e$ being 1 inch; so that $a\,d$ is the diagonal of a cube of 1 inch edge; draw $k\,f$ perpendicular to $a\,d$; in $a\,e$ produced set off $e\,m = m\,n = a\,e = 1$ inch: draw $n\,f$, $m\,g$, $e\,l$ perpendicular to $k\,f$: $k\,l$, $l\,g$, $g\,f$ will clearly be the isometric projections of straight lines each one inch long. Divide $a\,e$ into twelve equal parts, and draw through the points of subdivision straight lines parallel to $a\,d$: the line $k\,f$ will thus be graduated so as to form a scale of $\frac{1}{12}$ if used for feet; or of $\frac{1}{1}$ if used for inches.

Ex. 2. To construct an isometric scale of $\frac{1}{50}$ to measure feet.

Let the scale represent 10 feet.

Then 50 : 10 :: $\overset{\text{in.}}{12}$: No. of inches in the length of scale.

$$\therefore \text{ length} = \frac{\overset{\text{in.}}{12} \times 10}{50} = \overset{\text{in.}}{2\cdot4}.$$

Draw a line L (Pl. II. Fig, 8) 2·4 inches long: divide it into five equal parts, each will show 2 feet: subdivide the first primary division into 4 equal parts, each will show $\frac{1}{2}$ foot. The line L is, therefore, a scale of $\frac{1}{50}$.

To find the isometric projection of this scale: draw a line M making with L an angle equal to the angle which $a\,e$ makes with $k\,f$ (Fig. 7), and proceed as in Ex. 1. These scales should be completed as shown in the diagrams; their use will be exemplified in the following Exercises.

Ex. 3. To draw the isometrical projection of a rectangular block of wood: out of which a circle has been cut, the centre of the circle coinciding with the intersection of the diagonals of one face of the block.

Dimensions of block, $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{10} \times \overset{\text{ft.}}{2\frac{1}{2}}$, Radius of circle, $\overset{\text{ft.}}{3}$. Scale, $\frac{1}{50}$.

Let $e\,h$, $e\,a$, $e\,f$ (Pl. II., Fig. 9) be the axes. Set off on these axes $e\,h$ and $e\,f$ each equal to 10 feet measured on the scale in Fig. 8: complete the parallelogram $e\,f\,g\,h$, this will be the projection of a face of the solid $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{10}$: make $e\,a$ equal to $2\frac{1}{2}$ ft. complete the parallelograms $e\,a\,b\,f$, $e\,a\,d\,h$, these will be the projections of two faces each $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{2\frac{1}{2}}$; and the projection is completed. The dotted lines $b\,c$, $d\,c$, $c\,g$ are the projections of edges not visible.

To draw the projection of the circular aperture.

Let the diagonals $h\,f$ and $g\,e$ intersect in x: set off $x\,m$ equal to $x\,k$, equal to 3 feet, taken from the same scale as before: complete the rhombus $k\,l\,m\,n$: this will be the projection of a square of 6 feet side: it is required to construct the projection of the circle inscribed in this square, which will be an ellipse inscribed in the rhombus $k\,l\,m\,n$.

It is evident that o, p, q, r, the middle points of $n\,k, k\,l, l\,m, m\,n$,

will be four points in this curve: its major axis will be situated in $h\,f$, and, being the projection of a straight line parallel to the plane of projection, will be equal to the diameter of the circle. If, therefore, $x\,s$ be made equal to $x\,t$, equal to 3 feet taken from the line L in the scale Fig. 8: s and t will be the extremities of the axis major.

The minor axis bisecting the major at right angles must coincide with $e\,g$: its extremities, u and v, will be determined by drawing $t\,u$, $s\,u$ parallel to $h\,g$, $f\,g$, and meeting $e\,g$ in u; and $t\,v$, $s\,v$ parallel to $h\,e$, $f\,e$, meeting $e\,g$ in v: the figure $v\,s\,t\,u$ being the projection of a square inscribed in the circle. Eight points in the ellipse being thus determined the curve $c\,n$ be traced through them.

The projection of the circle in the face $a\,d\,m\,b$ may be obtained by a similar construction.

Fig. 10 is the isometric projection of a wooden tray 1 ft. 8 in. long, $11\frac{3}{4}$ in. broad, $5\frac{3}{4}$ in. deep, of material $1\frac{1}{3}$ in. thick; on a scale of $\frac{1}{12}$.

Fig. 11 is the isometric projection of a cylinder: height 7·5 feet, radius of base 2 feet. Scale $\frac{1}{50}$.

These have been inserted as examples of the applicability of this kind of projection; though it has not been deemed necessary to append any explanation of the construction.

EXERCISES.

1. Give the ratio of a straight line to its isometric projection, and prove its correctness by a diagram.

Draw the isometrical projections of the following objects:—

2. A rectangular parallelopiped 12 ft. × 10 ft. × 8 ft. Scale $\frac{1}{50}$.

3. A flight of 10 steps, each 8 ft. long, 1 ft. wide, and 8 in. deep. Scale $\frac{1}{24}$.

4. A table, 4 ft. long, $2\frac{1}{2}$ ft. wide, and $2\frac{1}{2}$ ft. high, the top 2 inches thick, the legs 2 in. square, and fixed $1\frac{1}{2}$ in. from the

outside of the table, with a circular hole $1\frac{1}{2}$ ft. in diameter in the middle of the top. Scale $\frac{1}{12}$.

5. A hexagonal right prism, height 9 ft., side of base 2 ft. Scale $\frac{1}{24}$.

6. A table 4 ft. long, $2\frac{1}{2}$ ft. wide, and 3 ft. high, top 3 in. thick, legs 2 in. square. Scale $\frac{1}{12}$.

7. A circle $2\frac{1}{2}$ in. in diameter.

8. A cylindrical box, made of $\frac{1}{2}$ in. deal, the exterior diameter being 8 inches and the height 3 inches. Scale $\frac{1}{2}$.

9. A rectangular tray, 3 in. long, 2 in. wide, 1 in. high, and the sides $\frac{1}{4}$ in. thick.

10. A box 2 ft. square, and $1\frac{1}{2}$ ft. deep, made of $\frac{1}{2}$ in. board, and having a circular hole 4 in. in diameter in each side and end. Scale $\frac{1}{4}$.

11. A piece of timber, 3 ft. long, $1\frac{1}{2}$ ft. wide, 3 in. thick, a hole, in the shape of the frustum of a cone, is bored through the thickness in the centre of the length and breadth. The upper diameter of the hole is 1 ft. the lower diameter 4 in. Scale $\frac{1}{8}$.

12. The walls of a cottage, from which the roof has been removed, consisting of one room, of which the external dimensions are, length 15 ft., breadth 12 ft., height 10 ft. 6 in. In one of the long sides are two windows with semicircular heads, each 5 ft. 6 in. high to the springing of the arch, 3 ft. 6 in. wide, 2 ft. 6 in. from the ground. In one of the shorter sides is a doorway 3 ft. wide, 6 ft. 6 in. high, reached by two steps, each 3 ft. long, 1 ft. wide, and 6 in. high. The walls are 1 ft. thick. Scale $\frac{1}{48}$.

CHAPTER VII.

MISCELLANEOUS EXERCISES.

1. Draw a line inclined at 30° and making an angle of 45° with the vertical plane; and a plane inclined at 50° and making an angle 65° with the vertical plane to contain the line.

2. The horizontal trace of a plane makes an angle of 30° with the ground line, draw the vertical trace on the supposition that the two traces really contain an angle of 65°, thence determine the angles this plane makes with both planes of projection.

3. Draw a line A B, one inch long and inclined at 40°, through A draw a line perpendicular to A B, but inclined at 20°, and through B a line, also perpendicular to A B, but inclined at 30°.

4. Draw the plan and elevation of an equilateral triangle A B C of 2·5 inches side, when its plane is inclined at 60°, and the side A B at 35°; determine the inclinations of the sides A C, B C.

5. Draw the figured plan and elevation of a square A B C D of 2·5 inches side, when the corners A, B, C, are at 1·1, 2, 2·9 inches above the horizontal plane.

6. Draw the same square when its diagonal B C is inclined at 15°, and its diagonal A D at 38°.

7. Draw a regular pentagon of 1·5 inches side, the plane of which is inclined at 50°, when one diagonal is horizontal, and show its elevation on a plane parallel to another diagonal.

8. Determine the angle the plane of the pentagon of last question makes with the vertical plane of the elevation.

9. Show, by its plan and elevation, an octahedron of 3·25 inches edge, when the edge A B is horizontal, and the face A B C inclined at 20°.

10. Three spheres of 1·5, 1, ·5 inches radii lie on a horizontal plane, each sphere touching the other two; represent them by their plans, and determine a plane touching all three.

11. Show that the orthographic projections of the edges of a prism formed by the intersection of the faces are parallel straight lines.

12. Show that the orthographic projection of a perpendicular to a plane is at right angles to the trace of that plane.

13. The lateral planes of a parallelopiped are given by their traces; the base is in the horizontal plane; the height of the solid is 35 units: draw its plan and elevation. Unit ·1 inch.

14. Draw the plan of a tetrahedron of 4 inches' edge, when two sides of the base are inclined 5° and 19° respectively.

15. Draw the plan of a right pyramid with a square base, upon the plane of the latter: the faces are equilateral triangles with sides 15 feet each. Scale $4\frac{1}{2}$ feet to an inch. Add an elevation, and a shadow, the inclination of the rays being 41°.

16. Draw the plan and shadow (each on the plane of the base) of an oblique heptagonal prism, the axis of which makes an angle of 45° with the base. The height of the solid is 30 feet; the scale $\frac{1}{120}$; and the base is a regular heptagon of 13 feet side.

17. Make a sectional elevation on a plane passing through the upper extremity of the axis of the prism and inclined to the plan of that axis 45°.

18. The lines joining three points, which are respectively 1, $1\frac{1}{2}$, and $2\frac{1}{2}$ inches above a horizontal plane, form a triangle, each side of which measures 2 inches *in plan;* show how you determine the real form of this triangle and the inclination of the plane containing it.

19. Draw the plan of a right pyramid 2·3 inches high, in

such a position that one of the diagonals of its base, which is a square of 2 inches side, may be inclined at 13°, whilst the other is horizontal; and draw on it also two horizontal contour-lines at the levels of 1 and 2 inches below the apex.

20. A plane cuts the horizontal plane at an angle of 45°; a second plane, cutting the horizontal plane at an angle of 20°, cuts also the first plane. The horizontal traces of the two planes meet when produced at an angle of 15°. Determine the inclination of the line of intersection of the first and second planes, and the angle which it makes with the horizontal of each plane. Name the scale of units adopted.

21. What is meant by the trace and projection of a line, and by the trace and projection of a plane?

22. By what data can the position in space of a point, a straight line, and a plane be determined?

23. Draw an indefinite line inclined at 30°, and one plane, containing that line, inclined at 50°, another plane also containing the line but inclined at 65°, determine the angle contained by these two planes.

24. The *plans* of two lines contain an angle of 56°, one line A B is inclined at 40°, at what angle is the other line A C inclined, if the angle B A C is half a right angle?

25. Draw the plan and elevation of a cube of 3·25 inches edge, when the planes of its faces A B C D and A C E G are inclined at 55° and 69°,

Or when the edges A B, A C are inclined at 46° and 22°.

Or when the corners A, B, C, are at 11, 20, 29 units above the horizontal plane.

26. Express by its two projections, a point in each region of space and distant 20 units from the vertical, and 8·5 from the horizontal, plane. Also express by their figured plans 4 points, each distant 30 units from $x\,y$, two *in* the vertical plane and on opposite sides of the horizontal, and the other two *in* the horizontal plane, but upon opposite sides of the vertical.

27. Draw the traces of two planes inclined 35° and 54° respectively to the horizontal plane, and making a dihedral angle of 110° with each other.

28. Show that the sections of the upper and lower faces of a cube by a vertical plane will be parallel to each other.

29. If a straight line be parallel to a plane, every plane passing through the line will cut the first plane in a line parallel to the given one. Show this.

30. If a right prism or cylinder be cut by a plane perpendicular to its base, the section will be a rectangle. Prove this.

31. Construct the plan of a right pyramid which has an axis of 35 feet (scale $\frac{1}{108}$) and a regular nonagon of 10 feet side for its base, upon the plane of one of the faces of the solid. Add an elevation upon a plane inclined 34° to the plan of the axis. Draw also the "section" by a plane bisecting the axis and parallel to the elevation.

32. Define the terms "projecting surface," "projecting plane," "trace of a line," "plane of projection," and "section."

33. Draw the plan of a right prism with hexagonal bases on the plane of one of the latter: the sides of the hexagons are 4 feet each, and the height of the solid is 11 feet, scale $\frac{1}{36}$. Project the shadow on the plan by parallel rays inclined 36°. Make a sectional elevation on a plane bisecting two adjacent faces of the prism.

34. Represent by their "figured" plans the following lines:—

A B	30 units long—inclined at			30°.
C D	30	,,	,, ,,	60°.
E F	30	,,	,, ,,	90°.

35. Show by their "scales of slope" three planes;
One inclined at 30°; one inclined at 50°; the other inclined at 70°.

36. Show the vertical trace of each of these planes on a plane making an angle of 25°, with the horizontals of each.

37. Show by its *traces* a plane inclined at 40°, containing a line inclined at 20° (the line to be shown by its plan and elevation).

38. If a second plane inclined at 60° contains the same line, determine the angle the two planes would contain.

39. Draw the plan and elevation of a prism 30 units long, with a regular pentagon of 10 units side for its base.

Either when one edge of the base is horizontal, and the edges are inclined at 20°; or when one diagonal of the base is horizontal, and the edges are inclined at 50°.

40. Determine the two projections of 2 points, one of which is 12 units behind the vertical plane of projection and 6·5 units above the horizontal; and the other 15 units below the horizontal and 7 units in front of the vertical plane.

41. Determine by their contours, or by their traces, planes inclined 45° and 30° respectively to the horizontal.

42. Draw the plan of an equilateral triangle, side 35 units, figure the angles of the plan 12, 47, and 25, and determine the original triangle and the inclination of its sides to the horizontal plane.

43. A truncated pyramid on a square base of 1·5 inches side, and the top surface of 1 inch side, is 2·5 inches high. Draw a section on a line crossing the plan diagonally.

44. Draw the horizontal projection of the truncated pyramid in Ex. 43, when its base is inclined at an angle 25° to the horizon, and one of the edges of the base rests upon the horizontal plane.

45. Draw an elevation of the same solid when the edge which rests upon the horizontal plane makes an angle of 30° with the ground line.

THE END.

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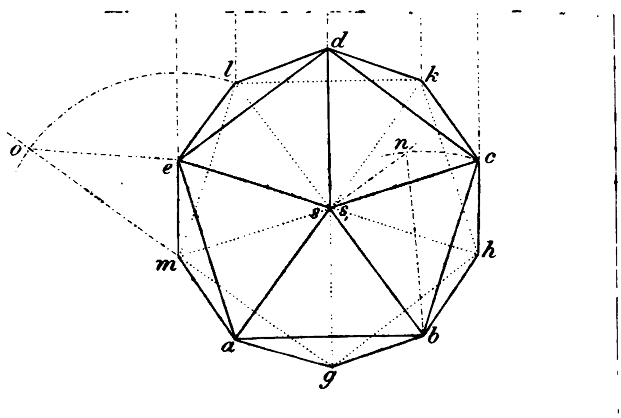

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PLATE II

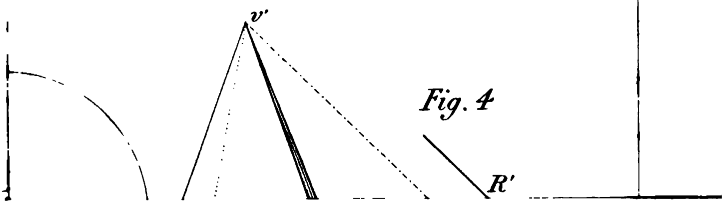

Fig. 4